SURVEY ON THE BURNSIDE RING OF COMPACT LIE GROUPS

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Abstract. The definition and basic properties of the Burnside ring of compact Lie groups are presented, with emphasis on the analogy with the construction of the Burnside ring of finite groups.

The Burnside ring of a finite group encodes the "calculus of cosets" of the group. It was defined by Burnside in his work on tables of marks of finite groups [3, p. 236]. The Burnside ring of a compact Lie group was defined by tom Dieck in the context of equivariant stable homotopy theory. It can also be described as an encoding of the "calculus of cosets", provided only certain transitive orbits of G are made visible. Namely, those transitive orbits, G/H, such that H has finite order in its normalizer. Some references are [1], [7], [9], [10], [16], [23, V].

1. The Burnside ring of a compact Lie group

First the definition of the Burnside ring of a finite group is given. Second the generalization of the Burnside ring from finite groups to compact Lie groups is presented. The compact Lie groups need not be connected, so all finite groups are compact Lie groups.

1.1. **The Burnside ring of a finite group.** Let G be a finite group. The Burnside ring of G is the Grothendieck group completion of the semiring of isomorphism classes of finite G-sets. It is denoted by $A(G)$ (other notations are $B(G)$ and $\Omega(G)$). Addition is given by disjoint union, and multiplication by Cartesian product. These operations are well defined on isomorphism classed of G-sets. The Burnside ring of G is isomorphic, as an abelian group, to the free abelian group with generators the isomorphism classes of transitive G-sets, G/H. Under this identification, the multiplication of the additive generators is given by the double coset formula. The double coset formula says that the G-set $G/H \times G/K$ is G-isomorphic to the disjoint union of the transitive G-sets $G/(H \cap gKg^{-1})$ where HgK runs over the double coset $H\backslash G/K$.

Let H be a subgroup of G and let X be a G-set. The H-fixed point set X^H is the subset $\{x \in X \mid hx = x, h \in H\}$ of X. The number of elements in X^H, denoted $|X^H|$, only depends on the G-isomorphism class of X and

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the G-conjugacy class of H. For every conjugacy class of a subgroup H of G the map

$$X \mapsto |X^H|$$

gives a semiring homomorphism from the semiring of isomorphism classes of finite G-sets to the integers. Hence there is an induced H-fixed point ring homomorphism $\phi_H\colon A(G) \to \mathbb{Z}$. The H-fixed point ring homomorphisms ensemble to give a ring homomorphism

$$\phi\colon A(G) \to \textstyle\prod_{(H)} \mathbb{Z},$$

where the product is over the G-conjugacy classes, (H), of subgroups H of G.

The map ϕ is sometimes called the mark homomorphism. Choose a linear ordering of the conjugacy classes of subgroups of G that respects subconjugacy. The matrix with the (H)-(K) entry given by $\phi_K([G/H])$ is called the table of marks, or the mark matrix, of G.

The basic properties of the Burnside ring of a finite group are described nicely in the first chapter of [10] and in [16]. A recent survey is [4].

1.2. **Recollections about compact Lie groups.** Let G be a compact Lie group (not necessarily connected). Only closed subgroups of G are considered. The Weyl group $W_G H$ of a subgroup H in G is $N_G H/H$, where $N_G H$ is the normalizer of H in G.

A theorem of Montgomery and Zippin says that for any closed subgroup H of G there is an open neighborhood U of the identity element in G such that all subgroups of HU are subconjugate to H [2, II.5.6],[26].

Let H and K be subgroups of G. The normalizer $N_G H$ acts from the left on $(G/K)^H$. Montgomery and Zippin's theorem implies that the coset $(G/K)^H/N_G H$ is finite [2, II.5.7]. In particular, if $W_G H$ is finite, then $(G/K)^H$ is finite. The Weyl group $W_G K$ acts freely on $(G/K)^H$ from the right by $gK \cdot nK = gnK$, where $gK \in (G/K)^H$ and $nK \in W_G K$. So $|W_G K|$ divides $|(G/K)^H|$. The fixed point space $(G/K)^H$ is nonempty if and only if H is subconjugated to K in G. Hence, if H is subconjugated to K and H has finite Weyl group, then K also has finite Weyl group.

A point x in a G-space X has orbit type J if the stabilizer subgroup $\{g \in G \mid gx = x\}$ of x is conjugate to J. Let $X_{(J)}$ denote the subspace of a G-space X consisting of the points of X with orbit type J. Let X_{fin} denote the subspace of a G-space X consisting of all points whose stabilizer has finite Weyl group. If H has finite Weyl group and X is a G-space, then $X^H_{\mathrm{fin}} = X^H$ by the results in the previous paragraph. A G-CW-complex is a space built out of n-dimensional G-cells $S^{n-1} \times G/H \to D^n \times G/H$, for subgroups H of G, by gluing them to cells of dimension $n-1$ or lower, for $n \geq 0$ (see [24, I.3] for a precise definition). If X is a G-CW-complex, then X_{fin} is a subcomplex of X. This follows because the stabilizer of any point in $D^n \times G/H$ is conjugated to H, and cells whose stabilizer have finite Weyl group can only map to other cells whose stabilizer also have finite Weyl group.

1.3. **Construction of the Burnside ring of a compact Lie group.** Let G be a compact Lie group. The basic idea in the definition of the Burnside ring of G is to consider finite disjoint unions of G-orbits, and ignore orbits, G/H, where H does not have finite Weyl group in G.

Denote the G-isomorphism class of a G-space X by $[X]$. The set of G-isomorphism classes of finite disjoint unions of transitive G-spaces, whose orbit types have finite Weyl group, has a structure of an abelian semigroup given by disjoint union.

If G is a compact Lie group, then the Cartesian product of two homogeneous G-spaces G/H and H/K is not isomorphic to a disjoint union of homogeneous G-spaces. So the definition of the product has to be modified. If G is a finite group, then a reformulation of the the double coset formula says that

$$[G/H][G/K] = \Sigma_{(J)} |(G/H \times G/K)_{(J)}/G| \, [G/J].$$

The following key observation is due to Schwäntzl [33].

Lemma 1.1. *Assume that Z is a G-space such that Z^J is a finite subspace of Z. Then $Z_{(J)}/G$ is a finite set.*

Proof. In fact there is an inequality $|Z_{(J)}/G| \leq |Z^J|$. It suffices to check this for G/gJg^{-1}. In this case the claim is true since $(G/gJg^{-1})/G$ is a point and $(G/gJg^{-1})^J$ is nonempty, for all $g \in G$. □

In particular, if X and Y are G-spaces so that X^J and Y^J are finite sets, then $(X \times Y)_{(J)}/G$ is a finite set. Hence, if J is a subgroup of G with finite Weyl group, and H and K are subgroups of G, then G/H^J and G/K^J are finite, and therefore $(G/H \times G/K)_{(J)}/G$ is finite.

Illman has proved that the product $G/H \times G/K$ is a finite G-CW-complex [22]. The G-cells of $G/H \times G/K$ with stabilizer a subgroup of G with finite Weyl group are all 0-dimensional by Lemma 1.1. Hence $(G/H \times G/K)_{\text{fin}}$, the subspace of $G/H \times G/K$ obtained by removing the G-cells whose stabilizers do not have finite Weyl group, is a finite disjoint union of homogeneous G-spaces.

Definition 1.2. Define a product as follows

$$[G/H] \cdot [G/K] = [(G/H \times G/K)_{\text{fin}}] = \Sigma_J \, n_J [G/J],$$

where the sum is over the conjugacy classes of subgroups J of G with finite Weyl group, and n_J is the number of elements in the finite set $(G/H \times G/K)_{(J)}/G$ [33].

The sum is finite since $G/H \times G/K$ has only finitely many orbit types. The isomorphism class of the point, $[G/G]$, is the multiplicative unit. The multiplication is clearly commutative and distributive with respect to the addition.

Lemma 1.3. *The multiplication in Definition 1.2 is associative.*

Proof. Consider three subgroups H, J, and K of G all with finite Weyl groups. It suffices to show that $((G/H \times G/J)_{\mathrm{fin}} \times G/K)_{\mathrm{fin}}$ is equivalent to $(G/H \times G/J \times G/K)_{\mathrm{fin}}$. Let U be a subgroup of G with infinite Weyl group. Then $G/U \times G/K$ consists of G-cells with stabilizers that are subconjugated to U. Hence they all have infinite Weyl groups (see section 1.2). The result follows. $\square$

Definition 1.4. Let G be a compact Lie group. Then the Burnside ring $A(G)$ is the Grothendick construction of the semiring of isomorphism classes of finite disjoint unions of homogeneous G-spaces, G/H, for subgroups H of G with finite Weyl group; the sum is given by disjoint union and multiplication is given by Definition 1.2.

Recall from section 1.2 that $(G/K)^H$ is finite, whenever H is a subgroup of G with finite Weyl group.

Lemma 1.5. *Let H be a subgroup of G with finite Weyl group. The function $\phi_H \colon A(G) \to \mathbb{Z}$, defined by $\phi_H(G/K) = |(G/K)^H|$ on the free generators $[G/K]$ of $A(G)$, is a ring homomorphism.*

Proof. It suffices to show that ϕ_H is a semiring homomorphism before passing to the Grothendieck construction. The map is well defined, additive and respects both the additive and the multiplicative units. The map respects the multiplication by observing that

$$(G/K \times G/L)^H_{\mathrm{fin}} = (G/K \times G/L)^H$$

since H has finite Weyl group. $\square$

1.4. **Other definitions of the Burnside ring.** There is a general categorical approach to Burnside rings [25]. May associates to any symmetric tensor triangulated category (with a skeletally small category of dualizable objects) a Burnside ring. It is a subring of the ring of selfmaps of the unit object. The Burnside ring of a compact Lie group is isomorphic to the Burnside ring associated to the G-equivariant stable homotopy theory (see section 8).

The following is a sketch of the main ideas in tom Dieck's original construction of $A(G)$ for compact Lie groups. The general categorical definition of May is modeled on this example. Instead of disjoint unions of homogeneous G-spaces the full subcategory of the stable homotopy category consisting of the dualizable objects is used. A spectrum is dualizable if and only if it is a suspension spectrum, $\Sigma^\infty_G X$, of a retract of finite G-cell complex X. There is a semiring of stable G-homotopy classes of such objects. (Alternatively, start out with all G-spaces of the homotopy type of a retract of a finite G-CW-complex. There is a semiring of G-homotopy types of such spaces; the sum is given by disjoint union, and the product by Cartesian product.) There is a semiring homomorphism into the integers given by Euler characteristic of the (geometric) H-fixed point spectra for subgroups H in G. These maps induces ring homomorphisms from the the Grothendieck construction of the semiring of stable G-homotopy classes of dualizable objects. The Burnside ring is the quotient ring given by dividing out the kernel of each of these homomorphism for all subgroups H of G. Hence a formal difference of two stable G-homotopy types $\Sigma_G^\infty X$ and $\Sigma_G^\infty Y$ is equal to 0 in the Burnside ring if and only if the Euler characteristic of the fixed point spaces X^H and Y^H are equal for all closed subgroups H in G. It is enough to check this for all subgroups H with finite Weyl groups since $\chi(X^K) = \chi(X^H)$, whenever H/K is a torus. The comparison of the definition of the Burnside ring sketched above to the one given in Definition 1.4 uses the additivity of the Euler characteristic on cofiber sequences.

The Burnside ring from the perspective of stable equivariant homotopy theory and geometric topology are surveyed in [24] [27].

1.5. **Maps between Burnside rings.** Let G_1 and G_2 be two compact Lie groups. Then there is a natural map

$$p\colon A(G_1) \otimes A(G_2) \to A(G_1 \times G_2)$$

given by sending G_1/H_1 and G_2/H_2 to $G_1 \times G_2/H_1 \times H_2$. This is well defined on isomorphism classes of transitive G-orbits. The map p is an injective ring map, however it is not an isomorphisms unless all subgroups of $G_1 \times G_2$, with finite Weyl groups, are of the form $H_1 \times H_2$, for subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$. If G_1 and G_2 are finite groups and $|G_1|$ and $|G_2|$ are relative prime, then p is an isomorphism.

Let G be a finite group. Then there is a map

$$\alpha\colon A(\mathbb{Z}/|G|) \to A(G)$$

such that for any subgroup H of G the composite map $\phi_H \circ \alpha$ is equivalent to $\phi_{\mathbb{Z}/|H|}$ where $\mathbb{Z}/|H|$ is the (unique) cyclic subgroup of $\mathbb{Z}/|G|$ of order $|H|$ [17].

Let H be a subgroup of a compact Lie group G. The induction map $\mathrm{ind}\colon A(H) \to A(G)$ is given by sending a generator $[H/K]$ to $[G/K]$ if K has finite Weyl group in G and to 0 otherwise.

The restriction map between Burnside rings is most easily described when the Burnside ring is defined in terms of equivalence classes of compact G-spaces (see section 1.4). Then the restriction map $\mathrm{res}\colon A(G) \to A(H)$ is defined by sending the isomorphism class of a G-space X to $X|H$, X regarded as an H-space via $H < G$. Let L be a subgroup of H with finite Weyl group in H, then

$$\phi_L \mathrm{res}\, x = \phi_{L'} x,$$

where L' is an extension in G of L by a torus such that L' has finite Weyl group in G. Let H be a normal subgroup of a finite group G. The restriction map $A(G) \to A(H)$ is give by sending the isomorphism class of a G-set G/L to

$$\frac{|G||H \cap L|}{|H||L|}\,[H/H \cap L],$$

for any subgroup L of G.

The restriction map $A(G) \to A(1) \cong \mathbb{Z}$ is called the augmentation map. The kernel of this map is called the augmentation ideal. The augmentation map is given by sending $[G/L]$ to $|(G/L)^T|$ where T is a maximal torus in G.

2. Examples of Burnside rings

2.1. **Burnside rings of abelian groups.** The only compact Lie groups with no proper subgroups with finite Weyl groups are the trivial group and the tori:

$$A(1) \cong \mathbb{Z}$$
$$A((S^1)^n) \cong \mathbb{Z},$$

for any $n \geq 1$. If G is a compact abelian Lie group, then G is isomorphic to the cartesian product of a torus and a finite abelian group. Hence if G is a compact abelian Lie group, then $A(G) \cong A(G/G^\circ)$, where G° is the unit component of G and G/G° is the group of components of G.

The multiplication in the Burnside ring (double coset formula) is particularly simple for finite abelian groups. It is given by

$$[G/K] \times [G/L] \cong \frac{|G|\,|K \cap L|}{|K|\,|L|}\,[G/K \cap L],$$

for subgroups K and L of G. While it is easy to find the isomorphism classes of subgroups of G, it is more involved to keep track of all subgroups of G, and their intersections.

The calculation of the Burnside ring of a compact abelian Lie group reduces to considering finite abelian p-groups, for primes p. They are of the form $G = \mathbb{Z}/p^{n_1} \times \cdots \times \mathbb{Z}/p^{n_m}$ where $n_i \geq 1$ and $m \geq 1$. In the rest of this subsection the bookkeeping of the subgroups is described in the special case of $\mathbb{Z}/p^n$ and elementary p-groups $(\mathbb{Z}/p)^n$. There is an isomorphisms

$$A(\mathbb{Z}/p) \cong \mathbb{Z}[x]/(x^2 - px).$$

More generally,

$$A(\mathbb{Z}/p^n) \cong \mathbb{Z}[a_1, \ldots, a_n]/a_i a_j = p^i a_j \quad (\text{for } j \geq i),$$

where a_i is the isomorphism class of the subgroup $\mathbb{Z}/p^i$. The Burnside ring $A((\mathbb{Z}/p)^2)$ is isomorphic to

$$\mathbb{Z}[a_0, \ldots, a_p, b]/a_i b = pb, b^2 = p^2 b, a_i^2 = pa_i, a_i a_j = b \text{ for } i \neq j.$$

The element b is $(\mathbb{Z}/p)^2$ and a_i is the subgroup of $(\mathbb{Z}/p)^2$ generated by the element $(1, i)$ for $0 \leq i < p$ and the subgroup generated by $(0, 1)$ when $i = p$.

To describe subgroups of $(\mathbb{Z}/p)^n$ the tactic is to associate to any subgroup H of $(\mathbb{Z}/p)^n$ a distinguished set of generators of H, reducing the problem to work with sets of generators instead of the subgroups themselves. This allow for a systematic description of the subgroups of G. The intersections of two subgroups given by two such sets of distinguished generators is the subgroup generated by the distinguished generators that are present in both sets of generators.

Fix n and set $G = (\mathbb{Z}/p)^n$. Let H be a subgroup of G. There is a tuple of integers (with $m \leq n$)

$$n \geq i_1 > i_2 > \cdots > i_m \geq 1$$

and elements $\alpha_1, \ldots, \alpha_m \in (\mathbb{Z}/p)^n$ such that $\alpha_k^{i_k} = 1$, $\alpha_k^j = 0$ for $j > i_k$, and $\alpha_k^{i_l} = 0$ whenever $l \neq k$. The superscript j denotes the jth coordinate in $(\mathbb{Z}/p)^n$. The subgroup H is the subgroup of G generated by the elements $\alpha_1, \ldots, \alpha_m$. The elements are linearly independent and H is isomorphic to $(\mathbb{Z}/p)^m$. There is exactly one such set of distinguished generators that generates H. It is illustrative to write the generators in the form of an $m \times n$-matrix with values in $\mathbb{Z}/p$. The following is an example when the rank of the subgroup is $m = 3$

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & * & \cdots . \end{pmatrix}$$

The intersection of a subgroup given by

$$n \geq i_1 > i_2 > \cdots > i_m \geq 1$$

and elements $\alpha'_1, \ldots, \alpha'_m \in (\mathbb{Z}/p)^n$ by another subgroup given by

$$n \geq i'_1 > i'_2 > \cdots > i'_{m'} \geq 1$$

and elements $\alpha'_1, \ldots, \alpha'_{m'} \in (\mathbb{Z}/p)^n$ is the subgroup described by the generators

$$n \geq j_1 > j_2 > \cdots > j_s \geq 1$$

and elements $\beta_1, \ldots, \beta_s \in (\mathbb{Z}/p)^n$ such that j_k is equal to both $i'_{j'}$ and i_j for some j and j' and such that $\alpha'_{j'} = \alpha_j$ for those integers; in this case $\beta_k = \alpha'_{j'} = \alpha_j$.

2.2. **Examples of Burnside rings of nonabelian groups.** If H is a normal subgroup of G with finite Weyl group, then $[G/H]^2 = |G/H|\,[G/H]$. More generally, if H and K are subgroups of G with finite Weyl groups, and H is normal in G, then

$$[G/H] \cdot [G/K] = \frac{|G/H|\,|(G/K)^{K \cap H}|}{|W_G(K \cap H)|}[G/K \cap H],$$

if $K \cap H$ have finite Weyl group in G, and $[G/H] \cdot [G/K] = 0$ if $K \cap H$ does not have finite Weyl group in G.

Let G be the permutation group Σ_3. It is isomorphic to the semidirect product $\mathbb{Z}/3 \rtimes \mathbb{Z}/2$. Let $a = [G/\mathbb{Z}/2]$, $b = [G/\mathbb{Z}/3]$, $c = [G/1]$, and $[G/G]$ is the identity element 1. Then the Burnside ring $A(G)$ is isomorphic to the polynomial ring $\mathbb{Z}[a, b, c]/\sim$, where the relations are $a^2 = a + c$, $b^2 = 2b$, $c^2 = 6c$, $ac = 3c$, $bc = 2c$, and $ab = c$.

Let G be the nontrivial semidirect product $S^1 \rtimes \mathbb{Z}/2$ (also known as O(2)). The subgroups of G with finite Weyl groups are G, $S^1 \rtimes 0$, and $\mathbb{Z}/n \rtimes \mathbb{Z}/2$ for $n \geq 1$. The normalizers are G, G, and $\mathbb{Z}/2n \rtimes \mathbb{Z}/2$ for $n \geq 1$, respectively. Let y denote the element $[G/S^1 \rtimes 0]$, and let x_n denote $[G/(\mathbb{Z}/n \rtimes \mathbb{Z}/2)]$, for $n \geq 1$. Let $\sim$ be the relations generated by $y \cdot y = 2y$, $x_n \cdot x_m = 2x_{(n,m)}$, for $m, n \geq 1$, and $x_n \cdot y = 0$, for $n \geq 1$, where (m, n) is the greatest common divisor of m and n. Then there is an isomorphism

$$A(S^1 \rtimes \mathbb{Z}/2) \cong \mathbb{Z}[y, x_1, x_2, x_3, ...]/\sim .$$

The Burnside ring of SU(3) is described in great detail in [10, 5.14].

Computer programs facilitate the calculation of the table of marks and the Burnside ring for many groups. See for example [28].

3. Rational description of the Burnside ring

3.1. **The space of closed subgroups.** Let $\mathcal{C}G$ be the space of closed subgroups of G with the Hausdorff topology from G. This is a compact metric space. Let ΨG be the quotient space of $\mathcal{C}$ obtained by identifying G-conjugate subgroups of G. The space ΨG is countable, hence it is a totally disconnected space [8].

Let ΦG be the subspace of ΨG consisting of conjugacy classes of closed subgroups of G with finite Weyl group. By Montgomery and Zippin's theorem the complement $\Psi G - \Phi G$ is open. So ΦG is a closed subspace of ΨG, hence a compact space.

There is a continuous retract map $\omega\colon \Psi G \to \Phi G$ given by sending (H) to the conjugacy class of a largest possible extension of H by a torus [21]. This extension is unique up to conjugation. The following result gives a useful description of ω [21, 2.2].

Lemma 3.1. *The conjugacy class $\omega(H)$ is the conjugacy class of the subgroup generated by (H) and a maximal torus in the centralizer $C_G H$.*

Let $C(G)$ denote the ring of continuous functions from ΦG to the integers $\mathbb{Z}$. Recall Lemma 1.5.

Lemma 3.2. *The homomorphisms ϕ_H, for $H \leq G$, ensemble to give a ring homomorphism*

$$\phi\colon A(G) \to C(G).$$

Proof. It remains to show that the map is continuous. It suffices to check that the map $(H) \mapsto |(G/K)^H|$ is continuous. Let (H_i) be a sequence converging to H. Montgomery and Zippin's theorem implies that there is no loss of generality in assuming that $H_i < H$. By cofinality, the sequence can be assumed to be such that $H_i < H_j$ for $i < j$. Hence $(H_i) \mapsto |(G/K)^{H_i}|$ is a decreasing function to the natural numbers, and hence it converges. The limit of the sequence is equal to $|(G/K)^H|$ because the closure of the union of the H_i is H. □

Lemma 3.3. *The mark homomorphism ϕ is an injective ring map.*

Proof. Assume that

$$\textstyle\sum_{i=1}^n q_i \, \phi([G/H_i]) = 0$$

where all $q_i \in \mathbb{Z}$ (or in $\mathbb{Q}$) are nonzero. Let (H_m) be a maximal conjugacy class among the $\{(H_i)\}$, for some m between 1 and n. The function evaluated at (H_m) is $q_m|W_G H_m| = 0$. This gives a contradiction. □

Let G be a finite group. Then the map ϕ is a surjective map only when $G = 1$.

The next subsection is devoted to prove that the inclusion map

$$\phi\colon A(G) \to C(G)$$

is an isomorphism after tensoring with the rational numbers $\mathbb{Q}$. The idea is to describe the topology on ΦG in a way which makes it clear that the functions $\phi([G/H])$ generate $C(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-module.

3.2. **The mark homomorphism ϕ is a rational isomorphism.** First a basis for the topology on ΦG is described.

Lemma 3.4. *The topology on ΦG is the smallest topology such that*

$$V(K) = \{(H) \in \Psi G \mid (H) \leq (K)\}$$

is both an open and a closed subset of ΦG for all $(K) \in \Phi G$.

Proof. The definition of the Hausdorff topology shows that $V(K)$ is closed. By Montgomery and Zippin's theorem $V(K)$ is also open for all $(K) \in \Phi G$.

Since ΦG is a countable metric space it has a basis for the topology consisting of open and closed sets (for each element x the function $d(x, -)$ is a continuous function to $\mathbb{R}$). Let (K) be in ΦG. Let U be an open and closed neighborhood of (K) in ΦG. Since $V(K)$ is open and closed there is no loss in generality assuming that U lies inside $V(K)$. The set $V(K) - U$ is open and closed. The collection of sets $V(H)$, for all (H) in $V(K) - U$, is an open covering of the closed subspace $V(K) - U$ of ΦG. Since ΦG is a compact space there is a finite set $\{(H_1), (H_2), \ldots, (H_n)\}$ such that each H_i is properly subconjugated to K in G and

$$V(K) - U \subset \cup_{i=1}^n V(H_i).$$

Then $V(K) - \cup_{i=1}^n V(H_i)$ is an open and closed neighborhood of (K) contained in U. Hence $V(K)$ and its complement $\Phi G - V(K)$, for $(K) \in \Phi G$, generates the topology on ΦG. □

Proposition 3.5. *The map*

$$\phi \otimes_{\mathbb{Z}} \mathbb{Q}\colon A(G) \otimes_{\mathbb{Z}} \mathbb{Q} \to C(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

Proof. It suffices to prove that the characteristic function on each of the open and closed subsets $V(H)$ for $(H) \in \Phi G$ are in the rational image of ϕ. Since $W_G H$ acts freely from the right on $(G/H)^K$ the values of $\phi(G/H)(K) = |G/H^K|$ are multiples of $|W_G H|$.

Let $V(H,n)$ be $\{(K) \mid |G/H^K| \geq n|W_G H|\}$. Then $V(H,1)$ equals $V(H)$. Define subsets

$$U(H,n) = \{(K) \mid |G/H^K| = n|W_G H|\} = V(H,n) - V(H,n+1).$$

Since $\phi(G/H)$ is a continuous function on a compact set, only finitely many of the $U(H,n)$ are nonempty. By considering linear combinations of different powers of $\phi(G/H)$ the characteristic function on $U(H,n)$ is in the rational image of ϕ. Hence so is the characteristic function on $V(H) = \cup_{n\geq 1} U(H,n)$. □

3.3. **An alternative additive basis of $C(G)$.** Assume that H has a finite Weyl group. Since $W_G H$ acts freely on $(G/H)^K$ it follows that $\phi_K([G/H])$ is divisible by $|W_G H|$ for all (K). Hence the function

$$a_H = \frac{1}{|W_G H|}\phi([G/H])$$

is in $C(G)$. It is not possible to divide any further since $a_H(H) = 1$.

Proposition 3.6. *The elements a_H, for $(H) \in \Phi G$, are linearly independent and generates $C(G)$ as an abelian group.*

Proof. The a_H are linearly independent. This follows from the proof of Lemma 3.3.

Any element f in $C(G)$ is in the rational image of ϕ by Proposition 3.5. It suffices to show that if

$$f = \textstyle\sum_{i=1}^{n} q_i\, a_{H_i}$$

is in $C(G)$, where $q_i \in \mathbb{Q}$, then $q_i \in \mathbb{Z}$, for all i. Assume that (H_k) is maximal among the (H_i) with $q_i \notin \mathbb{Z}$. Then $f(H_k) = q_k + \sum_j q_j a_{H_j}(H_k)$ where the sum is over subgroups H_j that properly contains a conjugate of H_k. This gives a contradiction, so all q_i are integers. □

3.4. **Congruence relations describing the image $\phi(A(G))$ in $C(G)$.** It is possible to describe the image $\phi(A(G))$ in $C(G)$ by a set of congruence relations. There is one congruence relation for each element in ΦG.

Lemma 3.7. *Let G be a finite group. Then*

$$\textstyle\sum_{g\in G} |X^g| \equiv 0 \ mod\ |G|$$

for all finite G-sets X.

Proof. Note that $\Sigma_{g\in G}\,|(G/H)^g|$ equals $\Sigma_{kH}\,|kHk^{-1}| = |G|$ by rearranging the summation. This implies that $\Sigma_{g\in G}\,|X^g| = |G||X/G|$. □

Proposition 3.8. *Let G be a compact Lie group. An element $f \in C(G)$ is in the image of $\phi\colon A(G) \to C(G)$ if and only if, for each $(H) \in \Phi G$, it satisfies the following congruence relation*

$$\textstyle\sum_C n_{C/H} f(C) \equiv 0 \ mod\ |W_G H|$$

where the sum is over all C such that $H \lhd C$ and C/H is a cyclic group; $n_{C/H}$ is the number of generators of the cyclic subgroup C/H.

Proof. For each $(H) \in \Phi G$ apply Lemma 3.7 to $N_G H/H$ acting on the finite $N_G H/H$-sets $(G/L)^H$ for any subgroup L of G with finite Weyl group. The congruence relation for H is

$$\textstyle\sum_C n_{C/H}\phi_C([G/L]) \equiv 0 \text{ mod } |W_G H|$$

for any $[G/L]$ where the sum is over all $H \lhd C$ such that C/H is a cyclic group and $n_{C/H}$ is the number of different generators of the cyclic subgroup C/H. Hence any element in the image of the mark homomorphism satisfies these relations.

Any element in $C(G)$ can be written as a sum $\Sigma_K\, m_K a_K$ where m_K are integers and all but finitely many of them are zero. If the element $\Sigma_K\, m_K a_K$ satisfies all the congruence relations, then it suffices to show that for each (K) the integer m_K is divisible by $|W_G K|$, since $m_K a_K = \frac{m_K}{|W_G K|}\phi([G/K])$ is then in the image of ϕ. Assume that this is not the case and let (H) be maximal for which m_H is not divisible by $|W_G H|$. Then the congruence relation for (H) gives that $m_H \equiv 0 \text{ mod } |W_G H|$. This is a contradiction. Hence $|W_G K|$ must divide m_K for all K. □

It is a theorem of tom Dieck that there is a greatest common divisor of $|W_G H|$ for all subgroups H of G [8]. Let $|G|$ denote the greatest common divisor of $|W_G H|$ for all subgroups H of G with finite Weyl group. If G is a finite group, then $|G|$ equals the number of elements in G. The congruence relations gives the following.

Corollary 3.9. *There is an inclusion $|G|C(G) \subset \phi(A(G))$.*

For certain purposes the number $|G|$ serves as the order of the compact Lie group. A generalization of the Artin induction theorem to compact Lie groups make use of a whole family of different orders of compact Lie groups [19]. These orders all reduce to the number of elements in G when G is a finite group.

4. The prime ideal spectrum of $A(G)$

The ring map $\phi\colon A(G) \to C(G)$ is an integral extension since $C(G)$ is additively generated by idempotent elements. Hence by the going up theorem in commutative algebra

$$\operatorname{spec}\phi\colon \operatorname{spec} C(G) \to \operatorname{spec} A(G)$$

is surjective. That is, all the prime ideals of $A(G)$ are of the form $\phi^{-1}(P)$ for a prime ideal P in $C(G)$.

4.1. **The prime ideals of $C(G)$.** The prime ideals in $C(G)$ are all obtained by applying the following standard lemma to $X = \Phi G$ and $R = \mathbb{Z}$.

Lemma 4.1. *Let X be a totally disconnected compact Hausdorff space, and let R be a ring. Then there is an homeomorphism*

$$q\colon X \times spec\, R \to spec\, C(X, R)$$

given by sending x, P to the prime ideal

$$\{f \in C(X, R) \mid f(x) \in P\}.$$

Proof. (Outline of a proof.) Let c be an ideal in $C(X, R)$. Let

$$I(c, x) = \{f(x) \mid f \in c\}$$

be the ideal of all the values of the functions in c at x. If J is a prime ideal, then there is a unique x_J such that $1 \notin I(c, x_J)$. The map sending J to $(x_J, I(c, x_J))$ is an inverse to q. The two maps are continuous. □

4.2. **The prime ideals of $A(G)$.** The prime ideals of $A(G)$ are all of the form

$$q(H, 0) = \{x \in A(G) \mid \phi(x)(H) = 0\}$$
$$q(H, p) = \{x \in A(G) \mid \phi(x)(H) \in p\mathbb{Z}\}$$

for p any prime number and (H) in ΦG. This follows by pulling back the prime ideals of $C(G)$ along ϕ.

Note that $\phi([G/H])(H) = |W_G H|$ and $\phi([G/H])(K) = 0$, whenever $(K) \not\leq (H)$. Hence if $(K) \neq (H)$ in ΦG, then $q(K, 0) \neq q(H, 0)$.

If p is prime and y is a finite $\mathbb{Z}/p$-set, then $|y| - |y^{\mathbb{Z}/p}|$ is divisible by p. Hence if $K \triangleleft H$ is a normal subgroup with a p-group quotient, then

$$\phi(x)(H) \equiv \phi(x)(K) \bmod p$$

for all x in $A(G)$. Hence, if $K \triangleleft H$ is a normal subgroup with a p-group quotient, then $q(K, p) = q(H, p)$. Given a closed subgroup H of G. There is a smallest normal closed subgroup H'_p of H such that H/H'_p is a p-group. This group might not have finite Weyl group as a subgroup of G. The conjugacy class $\omega(H'_p) = (H_p)$ with finite Weyl group associated to (H'_p) is called the p-perfection of H in G. There is also a largest conjugacy class (H^p) of an extension H^p of H in G such that H^p/H is an extension of a torus by a p-group and H^p has finite Weyl group. It follows that $|W_G H^p|$ is relative prime to p. If H has finite Weyl group, then H^p is a finite extension of H by a p-group and H^p is unique. There are identities

$$(H_p)^p = (H^p) \text{ and } (H^p)_p = (H_p),$$

and hence identities

$$q(H, p) = q(H^p, p) = q(H_p, p),$$

for all prime numbers p and all closed subgroups H in G. If $(H^p) \not\leq (K^p)$, then $\phi([G/K^p]) \in q(H^p, p)$ and $\phi([G/K^p]) \notin q(K^p, p)$, since $|W_G K^p|$ is not divisible by p. So $q(K, p)$ must be different from $q(H, p)$. A closed subgroup of G can not be properly subconjugated to itself. Hence $q(K, p) = q(H, p)$ if and only if $(H^p) = (K^p)$, or equivalently that $(H_p) = (K_p)$.

There is an inclusion $q(H, 0) < q(K, p)$ if and only if $(H_p) \leq (K) \leq (H^p)$, or equivalently that $(H_p) = (K_p)$. The prime ideals $q(H, p)$ for $(H) \in \Phi G$ and p a prime are maximal ideals. The prime ideals $q(H, 0)$ for $(H) \in \Phi G$ are minimal prime ideals.

The ring $A(G)$ has Krull dimension 1 for all compact Lie groups G.

Lemma 4.2. *Let $H \leq J \leq K$ be subgroups of G. If $\omega(H) = \omega(K)$, then $\omega(H) = \omega(J)$.*

Proof. The conjugacy class $\omega(H)$ is the conjugacy class of HT_H where T_H is a maximal torus in $C_G H$. Similarly for J and K. Since $C_G(K) \leq C_G(J) \leq C_G H$ the maximal tori can be chosen such that $T_K \leq T_J \leq T_H$. The torus T_H is a maximal torus in $C_G(HT_H)$ and T_K is a maximal torus in $C_G(HT_K)$. The assumption that HT_H and KT_K are conjugate subgroups in G gives that that T_H and T_K are conjugate tori in G. Since $T_K \leq T_J \leq T_H$ this implies that $T_H = T_J = T_K$. The result now follows since $HT_H = JT_J = KT_K$. □

The next result was first proved by Bauer and May [1].

Proposition 4.3. *Let $H \leq J \leq K$ be subgroups of G and assume that $q(H, p) = q(K, p)$. Then there is an equality $q(J, p) = q(H, p)$.*

Proof. Assume that H is a subgroup of J. Let J'_p be the smallest normal subgroup of J such that J/J'_p is a p-group. Then $J'_p \cap H$ is a normal subgroup of H such that $H/J'_p \cap H \leq J/J'_p$ is a p-group. Hence $H'_p \leq J'_p \leq K'_p$ and Lemma 4.2 gives the result. □

The space ΦG has only finitely many elements if and only if the Weyl group of a maximal torus T of G acts trivially on T [10, 5.10.8]. Hence $A(G)$ is a Noetherian ring if and only if the Weyl group of a maximal torus T of G acts trivially on T.

5. Idempotents and units in $A(G)$

The idempotent elements in $A(G)$, and the idempotent elements in $A(G)$ with a set of primes inverted, have been completely described [9] [21] [36]. In particular, when G is a finite group, then $A(G)$ has no idempotent elements different from 0 and 1 if and only if G is a solvable group [10, 5.11.4]. This fact was emphasized in [16].

Let π be a collection of prime numbers. A group H is π-perfect if the group of components of H has no nontrivial solvable quotient π-group. Let $\Phi_\pi(G)$ be the subspace of ΦG consisting of conjugacy classes of π-perfect subgroups of G with finite Weyl group.

Let X be a topological space, and let $\Pi_0(X)$ denotes the space of components (with the quotient topology from X). Let R be a ring, and let $R_{(\pi)}$ denote the localization of R obtained by inverting all primes not in the set π. There is a map $\beta\colon \Phi_\pi(G) \to \Pi_0(\operatorname{spec} A(G)_{(\pi)})$ defined by sending (H) to the component of the prime ideal $q(H,0)$.

Proposition 5.1. *The map*

$$\beta\colon \Phi_\pi(G) \to \Pi_0(\mathit{spec}\, A(G)_{(\pi)})$$

is a homeomorphism.

There is a close connection between idempotent elements in $A(G)$ and unit elements in $A(G)$. The following is immediate from the embedding of $A(G)$ into $C(G)$. If e is an idempotent element in $A(G)$, then $2e-1$ is a unit element in $A(G)$. If $\phi(u)$ is a unit element in $A(G)$, then $\frac{\phi(u)+1}{2}$ is an idempotent element in $C(G)$. If G is a compact Lie group and $|G|$ is odd, then $\frac{u+1}{2}$ satisfies the congruence relations of Proposition 3.8, because both u and 1 satisfy the relations and $|W_GH|$ is not divisible by 2 for any $(H) \in \Phi G$. Proposition 3.8 gives that $\frac{u+1}{2}$ is in $A(G)$. Hence there is a bijection between idempotents elements and unit elements in $A(G)$ when G has odd order.

There is a homomorphism $R(G;\mathbb{R}) \to A(G)^\times$ given by sending a real representation V to the function $(-1)^{\dim V^H}$ [10, 5.5.9]. Tornehave has proved that this map is surjective when G is a finite 2-group [35].

6. The representation ring

Let k be a field. There is a canonical map

$$A(G) \to R(G;k)$$

from the Burnside ring of G to the representation ring of G with coefficients in k. The map is given by sending G/H to the alternating sum of the G-representations $H^i(G/H;k)$ [7]. The map $A(G) \to R(G;k)$ is neither injective nor surjective in general. However if P is a finite p-group, then $A(P) \to R(P;\mathbb{Q})$ is surjective [31].

The composition $A(G) \to R(G;\mathbb{R}) \to A(G)^\times$ is called the exponential map of the Burnside ring. This map is surjective if G is a 2-group with no subquotients isomorphic to the dihedral group of order 16 [18].

7. Modules over $A(G)$

Modules over $A(G)$ have been studied by tom Dieck and Petrie [14]. Much attention has been given to invertible modules over $A(G)$ [11] [12] [15]. These are closely related to stable homotopy representations. A homotopy representation is a retract of a finite G-CW complex X such that X^H is homotopy equivalent to $S^{n(H)}$ for some integer $n(H)$, for each subgroup H in G. A stable homotopy representation is the suspension spectrum of a homotopy representation. Stable homotopy representation are exactly the invertible objects in the stable equivariant homotopy category [12] [20]. The finite groups G such that there are only a finite number of finitely generated indecomposable $A(G)$-modules (which are free over the integers) are characterized by Reichenbach in [29].

8. The Burnside ring in equivariant stable homotopy theory

Let X be a finite G-CW-complex. The (categorial) Euler characteristic of X turns out to be the stable homotopy class

$$\Sigma^\infty_G S^0 \xrightarrow{\tau} \Sigma^\infty_G X_+ \xrightarrow{c} \Sigma^\infty_G S^0$$

where the first map is the transfer map and the last map is the collapse map that sends X to a point. This defines a homomorphism

$$\chi\colon A(G) \to \pi_0^G(\Sigma^\infty_G S^0) = \pi_0^G(S^0).$$

There is a degree homomorphism

$$d\colon \pi_0^G(S^0) \to C(G)$$

which sends a stable self map $h\colon \Sigma^\infty_G S^0 \to \Sigma^\infty_G S^0$ to the continuous function that map (K) to the degree of h^K (geometric fixed points) as a stable self map of $\Sigma^\infty S^0$. The composite $d \circ \chi$ equals ϕ.

Theorem 8.1. *The Euler characteristic map*

$$\chi\colon A(G) \to \pi_0^G(S^0)$$

is an isomorphism.

The injectivity of χ follows from the injectivity of ϕ. The surjectivity of χ require a more careful understanding of $\pi_0^G(S^0)$. It suffices to show that $\pi_0^G(S^0)$ is additively generated by maps of the form $\Sigma^\infty_G S^0 \xrightarrow{\eta} \Sigma^\infty_G X_+ \xrightarrow{c} \Sigma^\infty_G S^0$. This follows from the spectrum level Segal–tom Dieck splitting theorem [23, IV.9.3].

Another proof consists of using equivariant obstruction theory to show that d is injective and that every function in the image of d satisfies the congruence relations of Proposition 3.8. Since d is injective the map χ must be surjective. This is done in [10, Chap. 8].

A variation of Theorem 8.1 is the Segal conjecture, proved by Carlsson [5]. Let G be a finite group and let EG be a free contractible G-space. The stable G-homotopy classes of maps from $\Sigma^\infty_G EG_+$ to $\Sigma^\infty_G S^0$ is isomorphic to the completion of the Burnside ring at its augmentation ideal. This is a deep result of importance in homotopy theory.

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