T-MODEL STRUCTURES ON CHAIN COMPLEXES OF PRESHEAVES

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Abstract. For a sheaf of rings R on a Grothendieck site, a tensor model structure is constructed on the category of chain complexes of presheaves of R-modules. The corresponding homotopy category is equivalent to the derived category provided the topos of sheaves of sets has enough points. The construction of the model structure is modified to give a family of t-model structures. These t-model structures give rise to a family of t-structures on the derived category, including the perverse t-structures.

1. introduction

This paper has to main objectives. The first is to give a *tensor* model category such that the associated homotopy category is the derived category of chain complexes of sheaves with its derived tensor product for an arbitrary unital ringed space. This is achieved by using the category of chain complexes of *presheaves* of R-modules for a sheaf of rings R, rather than the category of chain complexes of sheaves of R-modules. We make our construction in the more general context of a unital ringed topos with enough points. Our model structure is Quillen equivalent to the injective model structure on the category of chain complexes of presheaves provided the topos has enough points. The injective model structure on the category of chain complexes of (pre) sheaves is not a tensor model category. For certain ringed spaces, Hovey has constructed a tensor model structure on the category of chain complexes of sheaves of R-modules [13]. See also Gillespie [9].

The second objective is to show that the perverse t-structures, on the derived category of chain complexes of sheaves of R-modules for a suitable ringed space, lift to t-model structures on the category of chain complexes of *presheaves* of R-modules. This lift of a t-structure to a t-model structure basically means that the t-structure comes from well controlled truncations at the chain complex level. The t-model structures is constructed by modifying the localization techniques used in the construction of our model structure on the category chain complexes of *presheaves* of R-modules. The t-structures we construct are known for one-point topoi and in many cases for ringed spaces. See for example [1], [14] and [17]. Our main concern is to show that these t-structures actually lift to t-model structures.

We now give a summary of the paper. In Section 2 the definition of the derived category is reviewed. In Section 3 the projective model structure is generalize to the category of chain complexes of presheaves of R-modules. The weak equivalences are presheaf homology-isomorphisms. In Section 4 the projective model structure

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is given a tensor structure. In Section 5 we define quasi-simplicial model structures and show that the projective model structure is quasi-simplicial. In Section 6 the projective model category is localized with respect to the stalkwise homology isomorphisms. The resulting model structure is called the stalkwise model structure. It is a tensor model structure. If the ringed topos has enough points, then the homotopy category of the stalkwise model category is equivalent to the derived category $\mathcal{D}_R$.

In Section 7 families of t-structures on $\mathcal{D}_R$ are constructed as t-model structures on the category of chain complexes of presheaves of R-modules. The t-structures we consider are "locally" the standard t-structures: One family of t-structures is constructed by shifting the standard t-structure by an integer value at each point of the topos. These t-structures generalize the perverse t-structures on $\mathcal{D}_R$. In Section 8 we give an explicit description of $(\mathcal{D}_R)_{\geq 0}$ and $(\mathcal{D}_R)_{\leq 0}$ associated to particularly well-behaved t-structures.

An Appendix recalls Bousfield's cardinality argument in a form suitable for our applications. This is needed in Section 7. We assume the reader is familiar with the fundamentals of model category theory. See for example [10, 12].

2. The derived category

Let $\boldsymbol{\mathcal{C}}$ be a (skeletally) small Grothendieck site. Let $\boldsymbol{\mathcal{E}}$ denote the category of sheaves of sets on $\mathcal{C}$, and let ***Pre*** denote the category of presheaves of sets on $\mathcal{C}$. Let $i\colon \mathcal{E} \to Pre$ denote the inclusion functor. It has a left adjoint, the sheafification functor. We denote the sheafification functor by $\boldsymbol{L^2}$ ($=L \circ L$), and the unit of the adjunction by $\eta\colon 1 \to i \circ L^2$ [2, II.3.0.5]. Assume that $\mathcal{E}$ has a set of isomorphism classes of points, and let $\mathbf{pt}(\boldsymbol{\mathcal{E}})$ denote this set.

Let $\boldsymbol{R}$ be a sheaf of rings on $\mathcal{C}$. Let $\boldsymbol{\mathcal{S}}$ denote the category of left R-modules in $\mathcal{E}$, and let $\boldsymbol{\mathcal{P}}$ denote the category of left iR-modules in Pre. The functor i induces an inclusion functor $i\colon \mathcal{S} \to \mathcal{P}$, and the functor L^2 induces a left adjoint sheafification functor $L^2\colon \mathcal{P} \to \mathcal{S}$. The inclusion functor $i\colon \mathcal{S} \to \mathcal{P}$ is sometimes not made explicit. Both $\mathcal{S}$ and $\mathcal{P}$ are abelian closed tensor categories with units R and iR, respectively.

For any object C in $\mathcal{C}$, let $\boldsymbol{R_C}$ denote the free R-module in $\mathcal{P}$ generated by C [2, IV.11.3.3]. There is a natural isomorphism

$$\mathcal{P}(R_C, X) \cong X(C). \tag{2.1}$$

Similarly, L^2R_C is the free R-module in $\mathcal{S}$ generated by C. Let $\bullet$ be the terminal object in $\mathcal{C}$. Then R is isomorphic to $R_\bullet$.

Definition 2.2. Let $\mathbf{ch}(\boldsymbol{\mathcal{P}})$ denote the category of chain complexes of presheaves of iR-modules on $\mathcal{C}$, and let $\mathbf{ch}(\boldsymbol{\mathcal{S}})$ denote the category of chain complexes of sheaves of R-modules on $\mathcal{C}$.

The categories $\mathrm{ch}(\mathcal{S})$ and $\mathrm{ch}(\mathcal{P})$ are abelian closed tensor categories. Let $\boldsymbol{H_n(X)}$ denote the n-th (presheaf) homology of the chain complex X.

Definition 2.3. A map $f\colon X \to Y$ in $\mathrm{ch}(\mathcal{P})$ is a **presheaf homology-isomorphism** if

$$H_n(f)\colon H_n(X) \to H_n(Y)$$

is an isomorphism, for each $n \in \mathbb{Z}$.

A map $f\colon X \to Y$ in $\mathrm{ch}(\mathcal{P})$ is a **sheaf homology-isomorphism** if the sheafification of the induced map on homology

$$L^2H_n(f)\colon L^2H_n(X) \to L^2H_n(Y)$$

is an isomorphism, for each $n \in \mathbb{Z}$.

Definition 2.4. A map f in $\mathrm{ch}(\mathcal{P})$ is a **stalkwise homology-isomorphism** if $(L^2f)_p$ is a homology-isomorphism of chain complexes of R_p-modules for all points p in $\mathcal{E}$.

Let $\boldsymbol{i}$ also denote the inclusion functor $i\colon \mathrm{ch}(\mathcal{S}) \to \mathrm{ch}(\mathcal{P})$. A map f in $\mathrm{ch}(\mathcal{S})$ is said to be a presheaf homology-isomorphisms, sheaf homology-isomorphism, or stalkwise homology-isomorphism if $i(f)$ has this property.

If $\mathcal{E}$ has enough points, then a map f in $\mathrm{ch}(\mathcal{P})$ is a stalkwise homology-isomorphism if and only if it is a sheaf homology-isomorphism [2, IV.6.4.1].

Definition 2.5. The localization of $\mathrm{ch}(\mathcal{S})$ with respect to the class of all sheaf homology-isomorphisms is called the **derived category** of chain complexes of sheaves of R-modules on $\mathcal{C}$. It is denoted by $\boldsymbol{\mathcal{D}_R}$.

In Section 6 a tensor model structure on $\mathrm{ch}(\mathcal{P})$ is constructed such that the weak equivalences are the stalkwise homology-isomorphisms. Its homotopy category is equivalent to the derived category provided $\mathcal{E}$ has enough points. We first describe a preliminary model structure on $\mathrm{ch}(\mathcal{P})$ with weak equivalences the smaller class of presheaf homology-isomorphisms.

3. The projective model structure

Let R be a sheaf of rings. We define the cofibrant generators.

Definition 3.1. Let $\boldsymbol{i_{C,n}}$ be the (vertical) map of chain complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R_C & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{=} & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & R_C & \xrightarrow{=} & R_C & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

where $C \in \mathcal{C}$ and the vertical identity map on R_C is in degree n. Let $\boldsymbol{I}$ be the set of all $i_{C,n}$ for $C \in \mathcal{C}$ and $n \in \mathbb{Z}$.

Let $\boldsymbol{j_{C,n}}$ be the map of chain complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & R_C & \xrightarrow{=} & R_C & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

where $C \in \mathcal{C}$ and the rightmost copy of R_C is in degree n. Let $\boldsymbol{J}$ be the set of all $j_{C,n}$ for $C \in \mathcal{C}$ and $n \in \mathbb{Z}$.

The following model structure on $\mathrm{ch}(\mathcal{P})$ is called the **projective model structure** on the category of chain complexes of presheaves of R-modules. Relative I-cell complexes are defined in [10, 10.5].

Theorem 3.2. *There is a proper cofibrantly generated model structure on* $ch(\mathcal{P})$ *with weak equivalences the presheaf homology-isomorphisms and cofibrations the retracts of relative* I*-cell complexes. The fibrations are the levelwise surjective maps of presheaves. A set of cofibrant generators is* I *and a set of acyclic cofibrant generators is* J*. These generators have small sources.*

A map of presheaves $f\colon X \to Y$ is levelwise surjective (injective) if $f_n(C)$ is surjective (injective) as a map of sets, for all $C \in \mathcal{C}$ and $n \in \mathbb{Z}$. A map is levelwise surjective (injective) if and only if f is epic (monic) in the category $\mathcal{P}$. The cofibrations are included in the class of levelwise injective maps.

Proof. It suffices to check that: (1) $\mathrm{inj}\,(I)$ is equal to $\mathrm{inj}\,(J)\cap W$ and (2) $\mathrm{proj}\,(\mathrm{inj}\,(J))$ is contained in the class of weak equivalences W [10, 11.3.1]. Recall that $\mathrm{inj}\,(I)$ is the collection of maps that have the right lifting property with respect to all the maps in I.

We first show that $\mathrm{inj}\, J$ is equal to the class of surjective maps, and that $\mathrm{inj}\, I$ is equal to the class of surjective maps that in addition are homology-isomorphisms. This gives (1) and the description of the fibrations.

A map from $j_{C,n-1}$ to $f\colon X \to Y$ is specified by $y \in Y_n(C)$, and a lift is given by an element $x \in X_n(C)$ such that $f_n(C)(x) = y$. Hence f has the right lifting property with respect to $j_{C,n-1}$ if and only if $f_n(C)$ is surjective.

We denote the n-cycles of Y at C, $\ker(Y_n(C) \to Y_{n-1}(C))$, by $Z(Y(C))_n$. Observe that a map $f\colon X \to Y$ has the right lifting property with respect to $i_{C,n}$ if and only if the canonical map from $X_{n+1}(C)$ to $W_n(C)$ in the pullback diagram

$$\begin{array}{ccc} W_n(C) & \longrightarrow & Y_{n+1}(C) \\ \downarrow & & \downarrow \\ Z(X(C))_n & \longrightarrow & Z(Y(C))_n \end{array}$$

is surjective.

Assume that f has the right lifting property with respect to $i_{C,n-1}$. Then

$$Z(X(C))_n \to Z(Y(C))_n$$

is surjective since $0 \times Z(Y(C))_n$ is contained in $W_{n-1}(C)$. Hence $H_n(f)(C)$ is surjective and $W_n(C) \to Y_{n+1}(C)$ is surjective. The right lifting property with respect to both $i_{C,n-1}$ and $i_{C,n}$ implies that $f_{n+1}(C)\colon X_{n+1}(C) \to Y_{n+1}(C)$ is surjective. So the induced map on boundaries

$$\mathrm{im}(X_{n+1}(C) \to X_n(C)) \to \mathrm{im}(Y_{n+1}(C) \to Y_n(C))$$

is surjective. Hence $H_n(f)(C)$ is bijective.

We now prove the converse claim. Assume that f is a homology-isomorphism of presheaves and $f_n(C)$ is surjective, for all $C \in \mathcal{C}$ and $n \in \mathbb{Z}$. Given an element $x \in Z(X(C))_n$ and an element $y \in Y_{n+1}(C)$ such that $d(y) = f_n(x)$. We need to show that the element $(x, y) \in W_n(C)$ comes from an element in $X_{n+1}(C)$. There exists an element $x' \in X_{n+1}(C)$ such that $dx' = x$ because $H_n(f)$ is injective. The element $f_{n+1}(C)(x') - y$ in $Y_{n+1}(C)$ is a cycle. Since $H_{n+1}(f)(C)$ is surjective there exists an element $x'' \in X_{n+1}(C)$ such that $dx'' = x$ and $f_{n+1}(C)(x'') = y$. This shows that the diagram lifts. So $i_{C,n}$ has the left lifting property with respect to f.

We now verify (2). Assume that $f\colon X \to Y$ is a map in $\mathrm{proj}\,(\mathrm{inj}\,(J))$. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{f,\,\mathrm{id}} & Y \oplus X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y \oplus 0} \\ Y & \xrightarrow{=} & Y. \end{array}$$

The rightmost vertical map is surjective. So by our assumption on f the diagram lifts. Hence $H_n(f)$ is injective for all n.

Now consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id},\,0} & X \oplus \mathrm{Tot}(Y \oplus Y) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \oplus g} \\ Y & \xrightarrow{=} & Y \end{array}$$

where $\mathrm{Tot}(Y \oplus Y)$ is the total complex associated to the double complex

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & Y_{n+1} & \longrightarrow & Y_n & \longrightarrow & Y_{n-1} & \longrightarrow & \cdots \\ \downarrow & & \downarrow{\scriptstyle (-\mathrm{id})^{n+1}} & & \downarrow{\scriptstyle (-\mathrm{id})^{n}} & & \downarrow{\scriptstyle (-\mathrm{id})^{n-1}} & & \downarrow \\ \cdots & \longrightarrow & Y_{n+1} & \longrightarrow & Y_n & \longrightarrow & Y_{n-1} & \longrightarrow & \cdots . \end{array}$$

The map $g\colon \mathrm{Tot}(Y \oplus Y) \to Y$ is given by the identity map on the upper copy of Y in the double complex and by 0 on the lower copy of Y. Hence the rightmost vertical map in the diagram is surjective. By our assumption on f there is a lift in the diagram. Since the homology of $\mathrm{Tot}(Y \oplus Y)$ is 0 we get that $H_n(f)$ is surjective for all n.

The verification of properness reduces to the category of chain complexes of $R(C)$-modules for each $C \in \mathcal{C}$. A diagram chase shows that the pushout of a homology-isomorphism along a levelwise injective map of chain complexes is again a homology-isomorphism. A simpler verification shows that the pullback of a homology-isomorphism along a levelwise surjective map of chain complexes is again a homology-isomorphism. Both pushouts and pullbacks in $\mathrm{ch}(\mathcal{P})$ are formed levelwise. Since the cofibrations are levelwise injective, and the fibrations are levelwise surjective, it follows that the model structure is proper.

The cofibrant generators are small since evaluation of presheaves at an object C of $\mathcal{C}$ commutes with direct sum. □

We refer to Hovey for an alternative description of the cofibrant objects and the cofibrations in the projective model structure [12, 2.3.6, 2.3.8-9]. Note that the isomorphism in Equation 2.1 shows that the presheaf R_C is a projective object in $\mathcal{P}$, for each object $C \in \mathcal{C}$. In fact all projective objects in $\mathcal{P}$ are retracts of direct sums of object of the form R_C, where $C \in \mathcal{C}$. The projective objects of $\mathrm{ch}(\mathcal{P})$ are retracts of J-cell complexes.

Remark 3.3. The projective model structure on $\mathrm{ch}(\mathcal{P})$ is a stable model structure. Hence its homotopy category is a triangulated category [12, 7.1]. This triangulated category is different from the derived category of chain complexes of sheaves of R-modules on the site $\mathcal{C}$.

Definition 3.4. The **unit interval complex**, denoted $\boldsymbol{U}$, consists of two copies of R in degree 0, and one copy of R in degree 1, the differential is the identity map on the first copy and minus the identity map on the second copy of R.

If X is a cofibrant object, then a cylinder object for X is given by $X \coprod X \to X \otimes U \to X$, where U is the unit interval complex. Hence we get the following.

Lemma 3.5. *If X is a cofibrant object and Y an arbitrary object in the projective model structure on ch($\mathcal{P}$), then Ho(ch($\mathcal{P}$))(X, Y) is isomorphic to the group of chain homotopy classes of (degree 0) chain maps from X to Y.*

Proof. This follows since all objects are fibrant in the projective model structure on ch($\mathcal{P}$). □

Remark 3.6. The projective model structure on ch($\mathcal{P}$) is well known. We sketch an alternative construction of the projective model structure that mesh better with other examples of model structures of this kind.

The category of presheaves on $\mathcal{C}$ with values in the category of chain complexes of abelian groups is an abelian tensor category. Let A be a monoid in this abelian category. Denote the category of A-modules by A-ch($\mathcal{P}$). Let ch($\mathbb{Z}$) have the projective model structure. The projective model structure on A-ch($\mathcal{P}$) is inherited from the set of right adjoint functors

$$A\text{-ch}(\mathcal{P}) \to \text{ch}(\mathbb{Z})$$

given by evaluating at a set of objects C, representing each isomorphism class in $\mathcal{C}$, and composing with the forgetful functor from $A(C)$-modules to $\mathbb{Z}$-modules.

A similar model structure on the category of presheaves of simplicial sets has been given by Blander [5]. See also Hollander's model structure on the category of stacks [11].

A more elaborate class of examples of model categories of this type are strict model structures on diagram spectra. Diagram spectra are most naturally studied in an enriched setting.

These examples of presheaves on a site, as well as ours, first become interesting after we suitably localize them. We consider localizations in Section 6.

4. Tensor structures

From now on we assume that R is a sheaf of commutative rings. The category ch($\mathcal{P}$) is a symmetric closed tensor category. Let $\otimes_R$, or simply $\otimes$, denote the tensor product in ch($\mathcal{P}$). Let F_R, or simply F, denote the internal hom functor in ch($\mathcal{P}$). Our discussion of tensor model categories follows [15]. The next Lemma says that all cofibrant objects in the projective model structure are flat chain complexes.

Lemma 4.1. *Let K be a cofibrant object in ch($\mathcal{P}$) and let $f \colon X \to Y$ be a weak equivalence. Then $K \otimes f$ is also a weak equivalence in ch($\mathcal{P}$).*

Proof. The complex K is a retract of an I-cell complex K'. Let $f \colon X \to Y$ be a map of presheaves. There is a natural isomorphism

$$(R_C \otimes_R Z)(D) = \textstyle\bigoplus_{\mathcal{C}(D,C)} Z(D) \tag{4.2}$$

for any presheaf Z and any two objects C and D in $\mathcal{C}$. Hence $K'(D)$ is a directed colimit of bounded below complex of free $R(D)$-modules. The tensor product of a homology-isomorphism with a bounded below complexes of free modules is again a homology-isomorphism [16, 3.2, 5.8]. Homology commutes with directed colimits. Hence $K' \otimes_R X \to K' \otimes_R Y$ is a presheaf homology-isomorphism. A retract of a homology-isomorphism is again a homology-isomorphism, so $K \otimes f$ is a presheaf homology-isomorphism. □

The unit object for the tensor product on $\mathrm{ch}(\mathcal{P})$ is the chain complex with a copy of R in degree 0. We denote this chain complex also by R.

Lemma 4.3. *The unit object R in ch($\mathcal{P}$) is cofibrant. More generally, R_C is cofibrant ch($\mathcal{P}$), for $C \in \mathcal{C}$.*

Proof. The map of chain complexes $0 \to R_C$ is a pushout of the cofibration $i_{C,-1}$ along the map to the zero chain complex. Hence it is a cofibration. □

Let $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ be two maps. Then the pushout-product map is the canonical map

$$M(f_1, f_2) \colon \mathrm{colim}\left(Y_1 \otimes X_2 \xleftarrow{f_1 \otimes 1} X_1 \otimes X_2 \xrightarrow{1 \otimes f_2} X_1 \otimes Y_2\right) \longrightarrow Y_1 \otimes Y_2.$$

Definition 4.4. A model category with a tensor structure satisfies the **pushout-product axiom** if $M(f_1, f_2)$ is a cofibration whenever f_1 and f_2 are cofibrations, and $M(f_1, f_2)$ is an acyclic cofibration if f_1 or f_2 in addition is a weak equivalence.

A model category satisfying the pushout-product axiom is said to be a **tensor model category** [15, 3.1].

Lemma 4.5. *The projective model structure on ch($\mathcal{P}$) is a tensor model structure.*

Proof. We need to show that the projective model structure on $\mathrm{ch}(\mathcal{P})$ satisfies the pushout-product axiom. The (acyclic) cofibrations are closed under retracts, transfinite directed compositions, and pushout [15, 3.5]. So it suffices to show that if f_1 and f_2 are maps in I, then $M(f_1, f_2)$ is a relative I-cell complex, and if f_1 is a map in I and f_2 is a map in J, then $M(f_1, f_2)$ is a relative J-cell complex. Note that $R_{C_1} \otimes R_{C_2}$ is isomorphic to $R_{C_1 \times C_2}$, for objects $C_1, C_2 \in \mathcal{C}$. Denote this object by R_{12} for brevity. We have that $M(i_{C_1,0}, i_{C_2,0})$ is the inclusion map

$$\left(\cdots \to 0 \to 0 \to R_{12} \oplus R_{12} \xrightarrow{a} R_{12} \to 0 \to \cdots\right) \longrightarrow$$
$$\left(\cdots \to 0 \to R_{12} \xrightarrow{b} R_{12} \oplus R_{12} \xrightarrow{a} R_{12} \to 0 \to \cdots\right)$$

where a is the folding map and b is given by the identity map on the first factor and minus the identity map on the second factor. This is a relative I-cell complex. Similarly, the map $M(i_{C_1,0}, j_{C_2,0})$ is a relative J-cell complex. □

Definition 4.6. A model category with a tensor structure satisfies the **monoid axiom** if $j \otimes X$ is a weak equivalence for every acyclic cofibration j and any object X, and if any transfinite directed composition of pushouts of such maps is again a weak equivalence [15].

Lemma 4.7. *The category ch($\mathcal{P}$) with the projective tensor model structure satisfies the monoid axiom.*

Proof. This follows from Equation 4.2. □

Let A be a differential graded R-algebra of presheaves (i.e. a monoid in $\mathrm{ch}(\mathcal{P})$). Denote the category of A-modules in $\mathrm{ch}(\mathcal{P})$ by $\boldsymbol{A}$**-ch(**$\boldsymbol{\mathcal{P}}$**)**.

Lemma 4.8. *The category A-ch($\mathcal{P}$) inherits a cofibrantly generated model structure from ch($\mathcal{P}$) via the forgetful functor A-ch($\mathcal{P}$) $\to$ ch($\mathcal{P}$). If A is a symmetric monoid, then A-ch($\mathcal{P}$) is a tensor category (with tensor product over A) and the model structure on A-ch($\mathcal{P}$) satisfies the pushout-product axiom and the monoid axiom.*

Proof. This follows from Lemmas 4.1, 4.5, and 4.7 and [15, 4.1]. □

5. Quasi-simplicial model structures

We model theoretically enrich ch($\mathcal{P}$) in simplicial sets. We define a weakening of the axioms for a simplicial model category [10, 9.1.6]. For the definition of a simplicial category see [10, 9.1.2]. Let **Map** denote the (based) simplicial mapping space. Let $\boldsymbol{\mathcal{T}}$ denote the category of simplicial sets. Let $\boldsymbol{X \Box S}$ and $\boldsymbol{F_{\Box}(S, X)}$ denote the tensor and cotensor of $X \in \mathcal{K}$ by $S \in \mathcal{T}$, respectively.

Definition 5.1. A **quasi-simplicial** model category is a model category $\mathcal{K}$ which is a simplicial category satisfying the axioms below.

weakM6: Let X, Y be objects in $\mathcal{K}$ and let S be an object in $\mathcal{T}$. There is a natural isomorphism of simplicial sets

$$\mathrm{Map}(X \Box S, Y) \cong \mathrm{Map}(X, F_{\Box}(S, X)).$$

There is a natural isomorphism of *sets*

$$\mathcal{T}(S, \mathrm{Map}(X, Y)) \cong \mathcal{K}(X \Box S, Y).$$

M7: Let $i\colon A \to B$ be a cofibration in $\mathcal{K}$ and $f\colon X \to Y$ a fibration in $\mathcal{K}$. Then the map

$$i^* \times f_*\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a fibration of simplicial sets. If, in addition, i or f is a weak equivalence, then $i^* \times f_*$ is a weak equivalence.

Lemma 5.2. *Let $\mathcal{K}$ be a quasi-simplicial model category. Then the natural map $X \Box * \to X$ (corresponding to 1_X under the second adjunction in weakM6) is an isomorphism, for all $X \in \mathcal{K}$.*

Proof. The Yoneda lemma applied to the following composition of isomorphisms

$$\mathcal{K}(X, Y) \cong \mathrm{Map}(X, Y)_0 \cong \mathcal{T}(*, \mathrm{Map}(X, Y)) \cong \mathcal{K}(X \Box *, Y)$$

gives the result. □

Axiom M7 has an equivalent formulation in terms of the tensor or cotensor functors instead of the simplicial mapping space [10, 9.3.6]. One implication of axiom M7 is that $X \Box S \to X \Box T$ is a weak equivalence in $\mathcal{K}$ whenever X is cofibrant in $\mathcal{K}$ and $S \to T$ is an injective weak equivalence of simplicial sets. Combined with Lemma 5.2 this gives that the suspension of a cofibrant object X is equivalent to $X \Box S^1$ and the loop of a fibrant object Y is equivalent to $F_{\Box}(S^1, Y)$.

In a simplicial structure the second isomorphism of mapping sets in axiom weakM6 is strengthen to be an isomorphism of simplicial hom-sets. In a quasi-simplicial structure, unlike a simplicial structure, repeated applications of the tensor and cotensor functors need not respect the cartesian product (based: smash product) of simplicial sets [10, 9.1.11].

Lemma 5.3. *The projective model structures on ch($\mathcal{P}$) is quasi-simplicial.*

Proof. We make use of the Dold-Kan adjunction [18, 8.4]. Note that the inclusion functor, $\boldsymbol{k}$, from nonnegative chain complexes to unbounded chain complexes is left adjoint to the truncation functor, $\boldsymbol{\tau_{\geq 0}}$, given by sending

$$\cdots \to X_2 \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots$$

to

$$\cdots \to X_2 \to X_1 \to \ker(X_0 \to X_{-1}) \to 0 \to \cdots .$$

Let $\boldsymbol{D}$ denote the right adjoint of the normalized chain complex functor $\boldsymbol{N}$. Let S be a simplicial set. Let $R[S]$ be the free simplicial R-module, and let $NR[S]$ be the associated chain complex. We define the simplicial tensor to be $X \otimes_R kNR[S]$ and the simplicial cotensor to be $F(kNR[S], X)$, for $X \in \mathrm{ch}(\mathcal{P})$. The simplicial hom functor, $\mathrm{Map}(X, Y)$, is defined to be

$$Dj\tau_{\geq 0}\Gamma(F(X, Y))$$

for $X, Y \in \mathrm{ch}(\mathcal{P})$, where $\boldsymbol{\Gamma}$ denotes the global section functor, $\boldsymbol{j}$ denotes the forgetful functor from the category of ΓR-chain complexes to the category of chain complexes of abelian groups. The adjunctions in axiom weakM6 are satisfied.

Note first that the weakened version of axiom M6 still implies that axiom M7 has an alternative formulation as a pushout-product axiom in terms of the tensor [10, 9.3.7]. If $i\colon S \to S'$ is an inclusion of simplicial sets, then $kNR[i] : kNR[S] \to kNR[S']$ is a cofibration in the projective model structure, and if i is a weak equivalence, then $kNR[i]$ is a presheaf homology isomorphism in $\mathrm{ch}(\mathcal{P})$. Hence axiom M7 follows from Lemma 4.5. □

6. The stalkwise model structure

Given a (skeletally) small Grothendieck site $\mathcal{C}$. The following model structure is called the **stalkwise model structure** on $\mathrm{ch}(\mathcal{P})$.

Theorem 6.1. *There is a proper quasi-simplicial cofibrantly generated stable model structure on ch($\mathcal{P}$). The weak equivalences are stalkwise homology-isomorphisms and the cofibrations are retracts of relative I-cell complexes. With this model structure the tensor structure on ch($\mathcal{P}$) satisfies the pushout-product axiom and the monoid axiom.*

Proof. Let p be an arbitrary point in $\mathcal{E}$. The corresponding stalk functor from $\mathrm{ch}(\mathcal{P})$ to the category of chain complexes of R_p-modules respects both pushout and pullback squares, levelwise surjective maps, and levelwise injective maps. Furthermore, it takes homology-isomorphism of presheaves to homology-isomorphisms of R_p-modules.

For each point p in $\mathcal{E}$ the functor $(H_n)_p$ is a σ-uniform homology theory in $\mathrm{ch}(\mathcal{P})$ which commutes with arbitrary directed colimits (see Definitions A.2, A.3 and A.4). The homology groups are σ-uniform for a cardinal σ (for example the cardinality of the direct sum of $R(C)$ for all objects C in a skeleton of $\mathcal{C}$. Hence we can Bousfield localize $\mathrm{ch}(\mathcal{P})$ with respect to $(H_n)_p$-equivalences for $n \in \mathbb{Z}$ and p in $\mathrm{pt}(\mathcal{E})$. See Appendix A for details.

The model structure is proper and satisfies the pushout-product axiom and the monoid axiom. This follows from Theorem 3.2, Lemmas 4.5 and 4.7, and stalkwise verification of weak equivalences. The stalkwise model structure is quasi-simplicial with the same definitions of tensor, cotensor and simplicial hom functors as defined in Lemma 5.3. Axiom M7 follows because the stalkwise model structure satisfies the pushout-product axiom. The model category is stable. □

Let A be a commutative monoid in $\mathrm{ch}(R)$ (a commutative DGA). Then there are stalkwise model structures on the category of A-modules and A-algebras [15, 4.1]. (The cofibrant generators in the stalkwise model structure in Theorem 6.1 are small relative to the whole category for a suitable cardinal.)

The **injective model structure** on $\mathrm{ch}(\mathcal{S})$ has sheaf homology-isomorphisms as weak equivalences and levelwise injections as cofibrations. For a discussion of this model structure see [12, 2.3.13]. The fibrant objects are chain complexes of injective sheaves that are K-injective in the sense of Spaltenstein [16, 1.1].

Proposition 6.2. *Assume that the topos $\mathcal{E}$ has enough points. The adjoint pair (L^2, i) gives a Quillen equivalence between ch($\mathcal{P}$) with the stalkwise model structure and ch($\mathcal{S}$) with the injective model structure.*

In particular, the homotopy category of $\mathrm{ch}(\mathcal{P})$ with the stalkwise model structure is equivalent to $\mathcal{D}_R$ as a triangulated category.

Proof. The functor L^2 applied to a levelwise injective map of chain complexes of presheaves is a levelwise injective map of chain complexes of sheaves. Let X and Y be two objects in $\mathrm{ch}(\mathcal{P})$ and $\mathrm{ch}(\mathcal{S})$, respectively. The unit map $X \to i \circ L^2 X$ of the (L^2, i)-adjunction is a sheaf homology-isomorphism [2, II.3.2]. Hence a map $X \to i(Y)$ is a sheaf homology-isomorphism in $\mathrm{ch}(\mathcal{P})$ if and only if $L^2(X) \to Y$ is a sheaf homology-isomorphism in $\mathrm{ch}(\mathcal{S})$. Since the topos has enough points a map $X \to i(Y)$ is a sheaf homology-isomorphism if and only if it is a stalkwise homology-isomorphism, and i applied to a stalkwise homology-isomorphism is a sheaf homology. Hence the adjoint pair is a Quillen equivalence. □

Assume $\mathcal{E}$ has enough points. The class of fibrations for the stalkwise model structure on $\mathrm{ch}(\mathcal{P})$ contains the class of fibrations for the injective model structure on $\mathrm{ch}(\mathcal{S})$. The tensor product on $\mathrm{Ho}(\mathrm{ch}(\mathcal{P}))$ given by the tensor model structure on $\mathrm{ch}(\mathcal{P})$ gives the usual derived tensor product on $\mathcal{D}_R$ by Lemma 4.1. The homsets in the derived category $\mathcal{D}_R$ is described as in Lemma 3.5. The homset $\mathcal{D}_R(X, Y)$ is the set of homotopy classes of maps from a cofibrant replacement of X to a fibrant replacement of Y.

Proposition 6.3. *Assume that $i\colon \mathcal{S} \to \mathcal{P}$ respects direct sum. Then there is a proper quasi-simplicial cofibrantly generated model structure on ch($\mathcal{S}$) with cofibrant generators $L^2 I$. The weak equivalences are the stalkwise homology-isomorphisms and the cofibrations are retracts of relative $L^2 I$-cell complexes. The model structure satisfies the pushout-product axiom and the monoid axiom.*

Proof. The assumption implies that $L^2 R_C$ is small for all objects $C \in \mathcal{C}$. So the sources of $L^2 I$ and $L^2 J$ are small. Relative $L^2 J$-cell complexes in $\mathrm{ch}(\mathcal{S})$ are presheaf homology-isomorphisms. Hence there is a proper cofibrantly generated model structure on $\mathrm{ch}(\mathcal{S})$ such that the weak equivalences are presheaf homology-isomorphisms, and the cofibrant generators are $L^2 I$ and acyclic cofibrant generators are $L^2 J$ [10, 11.3.2]. Hence we can localize with respect to the stalkwise homology-isomorphisms as in Theorem 6.1 using Proposition A.12.

The model structure is proper, quasi-simplicial, and satisfies the pushout-product axiom and the monoid axiom. This is verified as in the proof of Theorem 6.1.

The quasi-simplicial structure constructed in the proof of Lemma 5.3 has to be modified by applying the sheafification functor to the tensor and cotensor functors constructed there. □

Proposition 6.4. *Assume that $i\colon \mathcal{S} \to \mathcal{P}$ respects direct sum. Let ch($\mathcal{S}$) and ch($\mathcal{P}$) both have the stalkwise model structure. The pair of adjoint functors $i\colon ch(\mathcal{S}) \to ch(\mathcal{P})$ and $L^2\colon ch(\mathcal{P}) \to ch(\mathcal{S})$ is a Quillen equivalence.*

Proof. The sheafification functor respects cofibrations and weak equivalences. A map $L^2X \to Y$ is a stalkwise homology-isomorphism if and only if the adjoint map $X \to i(Y)$ is a stalkwise homology-isomorphism for every $X \in \mathrm{ch}(\mathcal{P})$ and $Y \in \mathrm{ch}(\mathcal{S})$. □

If X is a Noetherian topological space and R is a sheaf of rings on X, then direct sums of sheaves in the category of presheaves are themselves sheaves. Hence the condition of Proposition 6.3 is satisfied.

Example 6.5 (Flat model structure)**.** Let (S, R) be a ringed space, and assume that (S, R) has finite hereditary global dimension [13, 3.1]. Under these assumptions Hovey has constructed a tensor model structure on $\mathrm{ch}(\mathcal{S})$, called the flat model structure [13, 3.2]. The assumptions are satisfied if S is a finite-dimensional Noetherian space [13, 3.3]. Hovey constructs a cofibrantly generated model structure on $\mathrm{ch}(\mathcal{S})$ with weak equivalences the stalkwise homology-isomorphisms. The fibrations are maps that are levelwise surjective, as presheaves, and whose kernels are complexes of flasque sheaves [13, 3.2].

We compare our model structure to his. Assume that (S, R) has finite hereditary global dimension. Let $\mathrm{ch}(\mathcal{P})$ have the stalkwise model structure and $\mathrm{ch}(\mathcal{S})$ the flat model structure. We claim that $L^2\colon \mathrm{ch}(\mathcal{P}) \to \mathrm{ch}(\mathcal{S})$ is a Quillen left adjoint functor to the forgetfull functor $i\colon \mathrm{ch}(\mathcal{S}) \to \mathrm{ch}(\mathcal{P})$. The functor L^2 respects cofibrations since L^2 respects colimits and L^2 of the cofibrant generators of the stalkwise model structure are among the cofibrant generators of the flat model structure [13, 1.1, 3.2]. In addition L^2 respects weak equivalences. So (L^2, j) is a Quillen adjunction. This is a Quillen equivalence since $L^2X \to Y$ is a sheaf homology-isomorphism in $\mathrm{ch}(\mathcal{S})$ if and only if the adjoint map $X \to i(Y)$ is a sheaf homology-isomorphism in $\mathrm{ch}(\mathcal{P})$, for $X \in \mathrm{ch}(\mathcal{P})$ and $Y \in \mathrm{ch}(\mathcal{S})$.

7. Some t-structures on derived categories

We construct t-structures on the homotopy category of $\mathrm{ch}(\mathcal{P})$ with the stalkwise model structure. These t-structures interact well with the model structure on $\mathrm{ch}(\mathcal{P})$. More precisely, they all arise from a t-model structure on $\mathrm{ch}(\mathcal{P})$. For the definition of a t-structure on a triangulated category see [7, 2.1]. The original source is [4]. Homological grading of t-structures is used. So $\mathcal{D}_{\geq n}, \mathcal{D}_{\leq n-1}$ corresponds to $\mathcal{D}^{\leq -n}, \mathcal{D}^{\geq -n+1}$ in cohomological notation. The heart of the t-structure is the intersection $\mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$.

We formulate the results of this section for stable model categories. Let $\boldsymbol{\mathcal{K}}$ be a stable model category. Then the homotopy category $\mathrm{Ho}(\mathcal{K})$ is a triangulated category [12, 7.1]. Assume a t-structure on $\mathrm{Ho}(\mathcal{K})$ is given. For the definition of homotopy fibers see [10, 13.4].

Definition 7.1. The class of ***n*-equivalences** in $\mathcal{K}$ is the class of maps f in $\mathcal{K}$ such that the homotopy type of $\mathrm{hofib}(f)$ is in $\mathrm{Ho}(\mathcal{K})_{\geq n}$, and the class of **co-*n*-equivalences** in $\mathcal{K}$ is the class of maps f in $\mathcal{K}$ such that the homotopy type of $\mathrm{hofib}(f)$ is in $\mathrm{Ho}(\mathcal{K})_{\leq n-1}$ [7, 3.1].

We now make precise what we mean by lifting a t-structure from the homotopy category $\mathrm{Ho}(\mathcal{K})$ to the model category $\mathcal{K}$.

Definition 7.2. A (weak) **t-model category** is a proper quasi-simplicial stable model category $\mathcal{K}$ with functorial factorization equipped with a t-structure on its homotopy category and a functorial factorization of maps in $\mathcal{K}$ as n-equivalences followed by co-n-equivalences, for each $n \in \mathbb{Z}$.

This is a weakening of the definition of a t-model structure given in [7, 4.1]. There the model structure is required to be simplical rather than quasi-simplicial. This is a harmless weakening and the results of [7] are still valid (with simplicial replaced by quasi-simplicial).

We are mainly interested in t-structures on the homotopy category of $\mathrm{ch}(\mathcal{P})$ with the stalkwise model structure, but we consider a more general framework. Let $\mathcal{K}$ be a proper quasi-simplicial stable cellular model category together with a t-structure on its homotopy category. Let I be a set of cofibrant generators of $\mathcal{K}$ [10, 12]. We make the following assumptions:

(1) the maps in I have small sources;
(2) the heart of the t-structure is the category of sheaves $\mathcal{S}$ (or of presheaves $\mathcal{P}$) of R-modules for a ring of sheaves R on a (skeletally) small Grothendieck site $\mathcal{C}$; and
(3) the heart functor, $\mathcal{H}\colon \mathcal{K} \to \mathcal{S}$ is a σ-uniform homology theory, for some cardinal σ (see Definition A.3).

The topos $\mathcal{E}$ is assumed to have a set, $\mathrm{pt}(\mathcal{E})$, of isomorphism classes of points. Let $\mathcal{D}$ denote the homotopy category of $\mathcal{K}$. Let

$$\mathbf{d}\colon \mathrm{pt}(\mathcal{E}) \to \mathbb{Z} \cup \{\pm\infty\}$$

be a function. We construct t-model structures on $\mathcal{K}$ by shifting the originally given t-structure on $\mathcal{K}$ in such a way that at each point of $\mathcal{E}$, in the isomorphism class $p \in \mathrm{pt}(\mathcal{E})$, the shift is given by $\Sigma^{d(p)}$. Let $\mathcal{H}_n$ denote the functor $\mathcal{H} \circ \Sigma^{-n}$. We make the convention that $\infty + n = \infty$ and $-\infty + n = -\infty$ for all integers n, and $\mathcal{H}_\infty(X) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n(X)$ and $\mathcal{H}_{-\infty}(X) = 0$.

Proposition 7.3. *There is a t-model structure (with simplicial relaxed to quasi-simplicial) on* $\mathcal{K}$ *such that*

$$\mathcal{D}_{\geq 0} = \{X \mid (\mathcal{H}_{n_p}(X))_p = 0 \textit{ for all } n_p < d(p)\}.$$

The associated class of n-equivalences, W_n, consists of all maps f such that $(\mathcal{H}_{n_p}(f))_p$ is an isomorphism for all $n_p < d(p)+n$ and $(\mathcal{H}_{d(p)+n}(f))_p$ is a surjection if $|d(p)| < \infty$.

Proof. A diagram chase shows that pushout of a W_n-map along a cofibration (levelwise injective map) is again a W_n-map (Lemma A.7). For each point p in the topos $\mathcal{E}$ the functor $(\mathcal{H})_p$ from $\mathcal{K}$ to abelian groups respects sums and directed colimits of relative I-cell complexes. We can localize the category $\mathcal{K}$ with respect to the $\mathbb{Z}$-indexed homology theory whose n-th functor is

$$X \mapsto \bigoplus_p (\mathcal{H}_{d(p)+n}(X))_p$$

using Proposition A.12. See also [7, 7.5]. □

The full subcategory $\mathcal{D}_{\leq -1}$ is

$$\{Y \in \mathcal{D} \mid \mathcal{D}(X, Y) = 0 \text{ for } X \in \mathcal{D}_{\geq 0}\}.$$

We describe $\mathcal{D}_{\leq -1}$ more explicitly in Section 8 when $\mathcal{K}$ is $\mathrm{ch}(\mathcal{P})$.

Corollary 7.4. *Let Z be a subset of $pt(\mathcal{E})$. Then there is a proper quasi-simplicial model structure on $\mathcal{K}$ such that the weak equivalences are $\bigoplus_{p \notin Z} (\mathcal{H}_*)_p$-isomorphisms and the cofibrations are retracts of relative I-cell complexes.*

Proof. Let $d_Z \colon \mathrm{pt}(\mathcal{E}) \to \mathbb{Z} \cup \{\pm\infty\}$ be the function defined by letting $d_Z(z) = \infty$, for $z \notin Z$, and $d_Z(z) = -\infty$, for $z \in Z$. The result follows from Proposition 7.3 applied to the function d_Z. □

The heart homology $\mathcal{H}_n(X)$ is a (pre)sheaf of R-modules. We refine the t-structure given in Proposition 7.3 by taking the structure of the ring R_p, for each point p, into account. Let

$$d \colon \coprod_{p \in \mathrm{pt}(\mathcal{E})} \operatorname{spec} R_p \to \mathbb{Z} \cup \{\pm\infty\}$$

be a function. We can localize the category $\mathcal{K}$ with respect to the $\mathbb{Z}$-indexed homology theory whose n-th functor is

$$X \mapsto \bigoplus_{p,\mathfrak{p}} ((\mathcal{H}_{d(p,\mathfrak{p})+n}(X))_p)_{\mathfrak{p}}.$$

Proposition 7.5. *There is a t-model structure (with simplicial relaxed to quasi-simplicial) on $\mathcal{K}$ such that*

$$\mathcal{D}_{\geq 0} = \{X \mid ((\mathcal{H}_{n_{p,\mathfrak{p}}}(f))_p)_{\mathfrak{p}} = 0 \textit{ for all } n_{p,\mathfrak{p}} < d(p,\mathfrak{p})\}.$$

The corresponding class of n-equivalences is the class of maps f in $\mathcal{K}$ such that

$$((\mathcal{H}_{n_{p,\mathfrak{p}}}(f))_p)_{\mathfrak{p}}$$

is an isomorphism for all $n_{p,\mathfrak{p}} < d(p,\mathfrak{p}) + n$, and a surjection for $n_{p,\mathfrak{p}} = d(p,\mathfrak{p}) + n$ if $|d(p,\mathfrak{p})| < \infty$.

Proof. The result follows from Proposition A.12. □

If d restricted to each $\operatorname{spec} R_p$ is a constant function, then Proposition 7.5 reduces to Proposition 7.3.

Remark 7.6. We can make further modifications to this construction for example by tensoring the heart homology functors with a flat module over the sheaf of rings R and localizing with respect to this homology theory instead. For example if $R \cong \mathbb{Z}$ (the one-point site), then we can use a rational heart homology theory $\mathcal{H} \otimes \mathbb{Q}$ instead of $\mathcal{H}$. If R is a sheaf of integral domains, then the sheafification of the presheaf of fraction fields $R_{(0)}$ is a flat R-module.

Example 7.7. The conditions on $\mathcal{K}$ required in this section are satisfied for the stalkwise model structure on the category of chain complexes of presheaves of R-modules, together with the standard t-structure on its homotopy category $\mathcal{D}_R$. The heart valued homology functor of the standard t-structure on $\mathcal{D}_R$, localized at a point p of the topos, is the usual homology of chain complexes localized at p. The homology functor, at each point, respects sums and directed colimits of relative I-cell complexes, and it is σ-uniform for some cardinal σ.

Let A be a monoid in $\mathrm{ch}(\mathcal{P})$ such that the sheaf valued homology is zero in negative degrees. The assumption on A assures that there is a standard t-structure on the homotopy category of A-$\mathrm{ch}(\mathcal{P})$ [7, Section 12]. Proposition 7.5 applies to the category of A-modules in $\mathrm{ch}(\mathcal{P})$ with the standard t-structure.

Example 7.8 (Ring)**.** Let $\mathcal{C}$ be the one morphism site. Then a ringed topos of presheaves on $\mathcal{C}$ is the category of sets together with a ring R, and $\mathcal{S} = \mathcal{P}$ is the category of R-modules. The projective and the stalkwise model structures on $\mathrm{ch}(\mathcal{S})$ coincide. The localization in Corollary 7.4 has been constructed by Neeman for the one morphism site [14, 3.3].

Let R be a Noetherian ring. Stanley has constructed the t-structures in Proposition 7.5 on the full subcategory of $\mathcal{D}_R$ consisting of complexes whose homology groups are finitely generated in each degree and bounded above and below [17]. He showed that there are no other t-structures on this full subcategory of $\mathcal{D}_R$ [17, 5.3].

Example 7.9 (Perverse t-structures)**.** We consider the category of R-modules for a ringed space (S, R). A topological space is said to be sober if all closed irreducible sets have a unique generic point. Let $k\colon S \to S_{\mathrm{sob}}$ be the universal map to a sober space. The points of S_{sob} corresponds to closed irreducible subsets of S, and the map k is given by sending a point $s \in S$ to the closure of s in S. The space of (isomorphism classes of) points of (S, R) is the space S_{sob} [2, IV.7.1.6].

We now assume that $S = S_{\mathrm{sob}}$ and compare our t-structures to the perverse t-structures introduced by Beilinson, Bernstein, Deligne and Gabber [4]. They consider a nonempty finite partition $\{S_a\}_{a\in A}$ of S into locally closed sets, together with a function $p\colon A \to \mathbb{Z}$, called the perversity function. A locally closed set is an intersection of an open and a closed set. The t-structure associated to a perversity function p is given by

$$\mathcal{D}_{\geq 0} = \{X \mid H_n(i_{S_a}^* X) = 0 \text{ for } n < p(a), a \in A\},$$

for the locally closed sets $\{S_a\}_{a\in A}$ in S [4, 2.2.1]. This agrees with

$$\{X \mid H_n(X)_q = 0 \text{ for } n < p(a), q \in S_a, a \in A\}.$$

Given a perversity function p. There is associated a function $d_p\colon S \to \mathbb{Z} \cup \{\pm\infty\}$, defined by letting

$$d_p^{-1}(n) = d_p^{-1}([n, \infty]) - d_p^{-1}([n+1, \infty]) = \textstyle\bigcup_{a\in p^{-1}(n)} S_a.$$

The perverse t-structure associated to p agrees with the t-structure in Proposition 7.3 for the associated function d_p. Hence the perverse t-structures on $\mathcal{D}_R$ lift to t-model structures on $\mathrm{ch}(\mathcal{P})$. If X is a Noetherian topological space, then the perverse t-structures on $\mathcal{D}_R$ lift to t-model structures on $\mathrm{ch}(\mathcal{S})$.

Example 7.10 (Stable homotopy category of spectra)**.** We give an application of Proposition 7.5 to a category which is not a derived category. The heart of the stable homotopy category of spectra $\mathcal{S}$ with the t-structure given by Postnikov sections is equivalent to the category of abelian groups. Hence Proposition 7.5 gives a twisted variant of the Postnikov t-structure. For each prime number p let $N_p \in \mathbb{Z} \cup \{\pm\infty\}$ and let $N_0 \in \mathbb{Z} \cup \{\pm\infty\}$. Then the associated full subcategory, $\mathcal{S}_{\geq 0}$ (the connective spectra), of $\mathcal{S}$ consists of spectra X such that

$$(H_{n_p}(X))_p = 0$$

whenever $n_p < N_p$, for all primes p, and $H_{n_0}(X) \otimes \mathbb{Q} = 0$ whenever $n_0 < N_0$.

Example 7.11 (Pro-chain complexes)**.** A t-model structure on $\mathrm{ch}(\mathcal{P})$ gives rise to a model structure on the category of pro-chain complexes of presheaves of R-modules. For example, there is a proper stable tensor model structure on pro-ch($\mathcal{P}$) such that the levelwise t-structure on the triangulated homotopy category of pro-ch($\mathcal{P}$) has the property that the intersection

$$\bigcap_{n\in\mathbb{Z}} \mathrm{Ho}\,(\text{pro-ch}(\mathcal{P}))_{\geq n}$$

consists of objects isomorphic to the zero object in Ho (pro-ch($\mathcal{P}$)) [7, 9.4]. It is essential to work with t-model structures for the construction of these triangulated categories. In fact examples like this was the motivation for introducing t-model structures. Work by the author on a general local cohomology theory makes use of model structures on categories of pro-chain complexes of presheaves of R-modules [8].

8. A description of $\mathcal{D}_{\leq 0}$ for the derived category

Recall that the homotopy category of $\mathrm{ch}(\mathcal{P})$ with the stalkwise model structure is equivalent to the derived category $\mathcal{D}_R$ provided $\mathcal{E}$ has enough points. We denote this category $\mathcal{D}$ for brevity. Let $d\colon \mathrm{pt}(\mathcal{E}) \to \mathbb{Z} \cup \{\pm\infty\}$ be a function such that $d^{-1}([n,\infty])$ is an open subset of $\mathrm{pt}(\mathcal{E})$, for every $n \in \mathbb{Z} \cup \{\pm\infty\}$. We describe the full subcategory $\mathcal{D}_{\leq 0}$ of $\mathcal{D}$ for the t-structure associated to d. We first recall some terminology and a Lemma.

The **support** of an object X in $\mathcal{E}$ is defined to be

$$\sup(X) = \{p \in \mathrm{pt}(\mathcal{E}) \mid X_p \neq \emptyset\}.$$

Note that if $X \to Y$ is a map in $\mathcal{E}$, then $\sup(X) \subset \sup(Y)$. Recall that the topology on $\mathrm{pt}(\mathcal{E})$ is generated by a basis of open sets consisting of $\sup(S)$, for all subobjects S of the terminal object $\bullet$ of $\mathcal{E}$ [2, IV.7.1.7]. Neighborhoods of points are defined in [2, IV.6.8].

Lemma 8.1. *Let X be an object in $\mathcal{E}$ and let p be a point of $\mathcal{E}$. Suppose given an element $x \in X_p$ and an open subset U of pt($\mathcal{E}$) containing p. Then there exists an object $C \in \mathcal{C}$ with support contained in U and a map $C \to X$ in $\mathcal{E}$ such that x is in the image of $C_p \to X_p$.*

In other words, p has a neighborhood with support contained in U.

Proof. By definition of the topology on $\mathrm{pt}(\mathcal{E})$ there is a subobject S of $\bullet$ such that $p \in \sup(S) \subset U$. The stalk, $S_p = \bullet$, is the colimit of $S(C)$ for neighborhoods $(C, c \in C_p)$ of p. Hence there exists a neighborhood $(C, c \in C_p)$ of p such that x in the image of $C_p \to X_p$ and $S(C) \neq \emptyset$ (so there is a map $C \to S$ in $\mathcal{E}$). Since $\sup(C) \subset \sup(S)$ the claim follows. □

Proposition 8.2. *Assume that $\mathcal{E}$ has enough points. Let $d\colon pt(\mathcal{E}) \to \mathbb{Z} \cup \{\pm\infty\}$ be a function such that $d^{-1}([n,\infty])$ is an open subset of $pt(\mathcal{E})$ for every element $n \in \mathbb{Z} \cup \{\pm\infty\}$. Then there is a t-model structure on ch($\mathcal{P}$) (with simplicial relaxed to quasi-simplicial) such that*

$$\mathcal{D}_{\geq 0} = \{X \mid (\mathcal{H}_{n_p}(X))_p = 0 \text{ for all } n_p < d(p)\}$$

and

$$\mathcal{D}_{\leq 0} = \{X \mid (\mathcal{H}_{n_p}(X))_p = 0 \text{ for all } n_p > d(p)\}.$$

Proof. By Proposition 7.3 there is a t-model structure on ch($\mathcal{P}$) such that $\mathcal{D}_{\geq 0}$ is the full subcategory above. The description of $\mathcal{D}_{\leq 0}$ follows by constructing the truncation functors associated to this t-structure explicitly.

For each object $C \in \mathcal{C}$ let $n(C) \in \mathbb{Z} \cup \{\pm\infty\}$ be the largest number such that $n(C) \leq d(p)$, for all $p \in \sup(C)$. Define a subcomplex $X_{\geq 0}$ of X as follows $(X_{\geq 0})_k(C)$ is $X_k(C)$ for $k > n(C)$, 0 for $k < n(C)$, and $\ker(X_{n(C)}(C) \to X_{n(C)-1}(C))$ for $k = n(C)$ if $|n(C)| < \infty$. There is a canonical inclusion map $X_{\geq 0} \to X$ and we denote the cokernel by $X_{\leq -1}$. This cokernel is weakly equivalent to the homotopy cofiber in the injective model structure on ch($\mathcal{P}$), since any injective map is a cofibration between cofibrant objects. Hence there is a distinguished triangle

$$X_{\geq 0} \to X \to X_{\leq -1} \to \Sigma X_{\geq 0}$$

in the homotopy category [12, 6.2.6, 6.3, 7.1]. The class of objects of the form $X_{\geq 0}$ are closed under suspension and the class of objects of the form $X_{\leq 0}$ are closed under desuspension. Since the transformations $X_{\geq 0} \to X$ and $Y \to Y_{\leq -1}$ are natural and $X_{\geq 0}$ and $Y_{\leq -1}$ both are the zero functor it follows that the derived hom sets $\mathcal{D}(X_{\geq 0}, Y_{\leq -1})$ are zero for all objects X and Y.

Lemma 8.1 implies that the map $X_{\geq 0} \to X$ is a stalkwise equivalence if and only if $X \in \mathcal{D}_{\geq 0}$, and $X \to X_{\leq -1}$ is a stalkwise homology isomorphism if and only if $X \in \mathcal{D}_{\leq -1}$. Since a t-structure is determined by $\mathcal{D}_{\leq 0}$ the result follows. □

Corollary 8.3. *The heart of the t-structure in Proposition 8.2 is given by*

$$\{X \mid (\mathcal{H}_{n_p}(X))_p = 0 \text{ for all } n_p \neq d(p)\}.$$

Remark 8.4. In Proposition 8.2 the description of $\mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}$ might be true under other assumptions on d. For example if $d^{-1}([n,\infty])$ is closed (instead of open), for all $n \in \mathbb{Z} \cup \{\pm\infty\}$, then Proposition 8.2 is still valid (define $n(C)$ to be the smallest number such that $n(C) \geq d(p)$, for all $p \in \sup(C)$).

Example 8.5 (Quasicoherent sheaves)**.** Let $(S, \mathcal{O}_S)$ be a scheme. There is a full abelian subcategory of $\mathcal{S}$ consisting of quasi-coherent $\mathcal{O}_S$-modules. Typically, $\mathcal{O}_U$ is not quasi-coherent for an open subset U of S. So we can not follow Chapter 3 and give a projective model structure on the category of chain complexes of quasi-coherent $\mathcal{O}_S$-modules.

We can construct t-model structures on the category of chain complexes of quasi-coherent presheaves using the techniques of Chapter 7 and a cofibrantly generated proper model structure given by Hovey for certain schemes [13, 2.4, 2.5]. The proof of Lemma 5.3 shows that his model structure is quasi-simplicial (the complex associated to a simplicial set are chain complexes of free $\mathcal{O}_S$-modules).

A t-structure on the derived category of quasi-coherent $\mathcal{O}_S$-modules can in some instances be inherited from a t-structure on $\mathcal{D}_{\mathcal{O}_S}$. Assume that S is a finite dimensional Noetherian scheme. Since S is quasi-compact and quasi-separated the derived category of chain complexes of quasi-coherent $\mathcal{O}_S$-modules is a full subcategory of the derived category of chain complexes of $\mathcal{O}_S$-modules [3, p.187]. Moreover, our assumptions guarantee that the objects of this full subcategory are exactly complexes with quasi-coherent homology [3, p.191].

Assume, furthermore, that the truncation $X_{\leq 0}$ has quasi-coherent homology whenever X has quasi-coherent homology. Then the t-structures on $\mathcal{D}_{\mathcal{O}_S}$, constructed in Proposition 7.5, clearly restrict to give t-structures on the derived category of chain complexes of quasi-coherent $\mathcal{O}_S$-modules. These perverse t-structures come from t-model structures: The t-model structures on the category of chain complexes of presheaves on S can be restricted to the full subcategory of chain complexes of presheaves with the weak homotopy type of a chain complex of sheaves of $\mathcal{O}_S$-modules with quasi-coherent homology. The homotopy category of this full subcategory is equivalent to the derived category of chain complexes of quasi-coherent $\mathcal{O}_S$-modules.

Example 8.6. Assume that the derived category of chain complexes of quasi-coherent R-modules is a full triangulated subcategory of $\mathcal{D}$. This is the case for the Zariski topology on a quasi-compact and quasi-separated scheme. The full subcategory inherit a t-structure from $\mathcal{D}$ if the truncation functors respect the full subcategory. In the next section conditions on the function $d\colon \operatorname{spec} R \to \mathbb{Z} \cup \infty$ are given which guarantee that truncation functors respect this full subcategory. [REFERENCES, objects are equivalent to q coherent if their homology is so. Look at older versions of this article.]

Note that many different functions $d\colon \operatorname{spec} R \to \mathbb{Z} \cup \infty$ will give rise to the same t-model structures on the full subcategory of $\mathcal{D}$ of chain complexes of quasi-coherent R-modules. For example if a point m is a specialization of a point p, then we might as well require that $d(p) \geq d(m)$.

Recall that a point m is a specialization of a point p if there is a map $p \to m$ [2, 4.2.2]. If M is a quasi-coherent module and $M_m = 0$, then $M_p = 0$. This follows from the corresponding property of the sheaf of rings R. Which again follows from considering the equalizer S of the maps 0 and 1 from the terminal object $\bullet$ to R. The subobject S of $\bullet$ has $S_P = \bullet$ if and only if $R_p = o$. For any subobject S of $\bullet$ there is a map $S_m \to S_p$. Hence if $R_m = 0$, then S_p. (This also shows that any open subset of the space of isomorphism classes of points of $\mathcal{E}$ that contains m must also contain p. Hence if m is a specialization of p, then m is in the closure of p.)

Appendix A. Bousfield's cardinality argument

We use Bousfield's cardinality argument to produce functorial factorizations [6]. Let $\mathcal{K}$ be a proper cellular stable model category with functorial factorizations [10, 12.1.1]. Let I be a set of cofibrant generators.

Definition A.1. Let X be an I-cell complex. The cardinality of the set of cells in X is denoted $\sharp X$. Let $i\colon A \to X$ be a relative I-cell complex. The cardinality of the set of I-cells used to build X from A is denoted $\boldsymbol{\sharp(X, A)}$.

Definition A.2. Let h be a functor from $\mathcal{K}$ to the category of sets. The functor h is said to satisfy the **colimit axiom** if for all relative I-cell complexes $A \to X$

$$\operatorname{colim}_\alpha h(X_\alpha) \to h(X)$$

is a bijection where the colimit is over all relative sub I-cell complexes $i_\alpha\colon A \to X_\alpha$ of i such that $\sharp(X_\alpha, A)$ is finite.

Definition A.3. Let σ be a cardinal. We say that a functor h from relative I-cell complexes to sets is ***σ*-uniform** if the cardinality of $h(X)$ is less than or equal to $\sigma \times \sharp X$ for all I-cell complexes X.

Definition A.4. A homology theory h (satisfying the colimit axiom) on the triangulated category $\mathrm{Ho}(\mathcal{K})$ is a functor h from $\mathrm{Ho}(\mathcal{K})$ to abelian groups such that distinguished triangles are sent to exact sequences, and h composed with the localization functor $\mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ satisfies the colimit axiom.

The exactness property of h implies that it respects finite sums. The colimit axiom furthermore implies that it respects arbitrary sums. If h is a homology theory, then $h \circ \Sigma^n$ is again a homology theory and it is denoted h_{-n}.

Definition A.5. Given a homology theory h. Let E be the class of all maps f such that $\bigoplus_{s<0} h_s(f)$ is injective and $\bigoplus_{s\leq 0} h_s(f)$ is surjective.

Note that the class E is closed under compositions and retracts but it need not satisfy the two-out-of-three property. If h is the heart homology functor for a t-structure, then E associated to h_{-n} is the class of n-equivalences (see Definition 7.1). In this case $\mathrm{inj}\,(E \cap C)$ is the class of fibrations that are co-n-equivalences [7, 4.6].

Lemma A.6. *The class E has the left cancelation property: If f and g are any two composable maps such that f and $g \circ f$ are in E, then g must also be in E.*

Proof. This follows since the class of surjective and bijective maps both have this property. □

Lemma A.7. *The class E is closed under homotopy pushouts.*

Proof. If the map $g\colon A' \to B'$ is the pushout of the map $f\colon A \to B$ along a cofibration, then the two horizontal sequences

$$\begin{array}{ccccccccc}
h_k(\Sigma^{-1}C) & \longrightarrow & h_k(A) & \xrightarrow{h_k(f)} & h_k(B) & \longrightarrow & h_k(C) & \longrightarrow & h_k(\Sigma A) \\
\cong\downarrow & & \downarrow & & \downarrow & & \cong\downarrow & & \downarrow \\
h_k(\Sigma^{-1}C') & \longrightarrow & h_k(A') & \xrightarrow{h(g)} & h_k(B') & \longrightarrow & h_k(C') & \longrightarrow & h_k(\Sigma A')
\end{array}$$

are exact. If $h_k(f)$ is injective, then a diagram chase shows that $h_k(g)$ is also injective. If $h_k(f)$ is surjective and $h_k(\Sigma f) = h_{k-1}(f)$ is injective, then $h_k(C)$ is the null-object, hence $h_k(g)$ is also surjective. □

We want to produce a functorial factorization of a map in $\mathcal{K}$ as a map in $C \cap E$ followed by a map that has the right lifting property with respect to all maps in $C \cap E$.

We say that $f\colon A \to X$ is an **inclusion of *I*-cell complexes** if A is an I-cell complex and f is a relative I-cell complex.

Definition A.8. Let $\boldsymbol{E}^{\mathbf{cell}}$ denote the class of all inclusions of I-cell complexes that are in E. Let $\boldsymbol{E}^{\mathbf{cell}}_{\boldsymbol{\sigma}}$ denote the class of all inclusions of I-cell complexes $A \to X$ in E such that $\sharp X \leq \sigma$. Let $\boldsymbol{E}^{\mathbf{rcell}}$ denote the class of all relative I-cell complex in E.

The class $E^{\mathrm{cell}}_{\sigma}$ is skeletally small for each cardinal σ, but E^{cell} and E^{rcell} need not be skeletally small. The next result is a variation of Bousfield's cardinality argument [10, 4.5.5].

Lemma A.9. *Let $\mathcal{K}$ be a cellular model category. Let σ be an infinite cardinal, and let h be a σ-uniform homology theory.*

Let $f\colon A \to X$ be an inclusion of I-cell complexes in the class E and with $A \neq X$. Then there is an I-cell subcomplex B of X such that $\sharp B \leq \sigma$, $B \not\subset A$, and $B \cap A \to B$ and $A \to A \cup B$ are in the class E.

Proof. We prove the following result: Let j and k be σ-uniform functors from $\mathcal{K}$ to the category of sets. Assume that $j(f)$ is injective and $k(f)$ is surjective. Then there is an I-cell subcomplex B of X such that $\sharp B \leq \sigma$, $B \not\subset A$, and $g\colon B \cap A \to B$ is an inclusion of I-cell complexes such that $j(g)$ is injective and $k(g)$ is surjective.

We construct an increasing sequence

$$B_0 \subset B_1 \subset B_2 \subset \cdots$$

of I-cell subcomplexes of X such that:

(1) $B_0 \not\subset A$.
(2) whenever two elements $i_1, i_2 \in j(B_n \cap A)$ map to the same element in $j(B_n)$, then they map to the same element in $j(B_{n+1} \cap A)$
(3) the set $k(B_n)$ maps onto the image of $k(B_{n+1} \cap A)$ in $k(B_{n+1})$.

We choose a finite subcomplex B_0 of X that is not contained in A. This is possible since $A \neq X$ and because the gluing map from any I-cell to A factors through a finite I-cell subcomplex of A, since the sources of the maps in I are small.

Assume that B_n has been constructed. We construct B_{n+1}. Let $i_1, i_2 \in j(B_n \cap A)$ be two elements which map to the same element in $j(B_n)$. By our assumption on f the two elements are sent to the same element under $j(B_n \cap A) \to j(A)$. The colimit axiom for j implies that there is a finite relative I-cell complex $B_n \to I_{x_1,x_2}$ in X such that the two elements map to the same element in $j(I_{x_1,x_2} \cap A)$.

Similarly, for every element $y \in k(B_n)$ there is a finite relative I-cell complex $B_n \to S_y$ in X such that the image of y in $k(S_y)$ is in the image of $k(S_y \cap A)$.

We make the following definition

$$B_{n+1} = B_n \cup_{i_1,i_2} I_{i_1,i_2} \cup_y S_y$$

where the sum is over $i_1, i_2 \in j(B_n \cap A)$ and $y \in k(B_n)$. This complex satisfies the conditions in the list above.

Let B be the union of all the B_n. The colimit axiom gives that $B \cap A \to B$ is in E_σ. The assumption that j and k are σ-uniform gives that $\sharp B \leq \sigma$.

The Lemma follows by letting j and k be the two functors given in Definition A.5. The inclusion of I-cell complexes $A \to A \cup B$ is the pushout of $B \cap A \to B$ along the cofibration $B \cap A \to A$. Hence the last claim follows from Lemma A.7. □

Lemma A.10. *The class injE^{cell} is equal to the class injE_σ^{cell}.*

Proof. Let $f\colon X \to Y$ be a map in E^{cell}. It suffices to show that f is a transfinite composition of maps in E_σ. Lemmas A.6 and A.9 gives a transfinite sequence of subcomplexes X_λ of Y containing X such that:

(1) $X_\lambda \to X_{\lambda+1}$ is in E_σ
(2) if λ is a limit ordinal, then $X_\lambda = \cup_{l<\lambda} X_l$
(3) if X_λ is strictly contained in X, then $X_{\lambda+1}$ is strictly larger than X_λ.

Zorn's Lemma implies that $X_\lambda = X$ for some λ [10, 4.5.6]. □

Lemma A.11. *The class of morphisms* $\mathrm{inj}\, E^{cell}$ *is equal to* $\mathrm{inj}\, E^{rcell}$.

Proof. The model category $\mathcal{K}$ is left proper. Hence this is a consequence of Proposition 13.2.1 in [10]. ☐

Proposition A.12. *Let* h *be a* σ*-uniform homology theory for some cardinal* σ*. Assume* $\mathcal{K}$ *is a proper cellular stable model category. Then there is a functorial factorization of any map in* $\mathcal{K}$ *as a relative* E_σ*-cell complex followed by a map in* $\mathrm{inj}\,(E \cap C)$*. Relative* E_σ*-cell complexes are contained in* $E \cap C$*. The class of maps* $E \cap \mathrm{inj}\,(E \cap C)$ *equals* $\mathrm{inj}\, C = W \cap F$*.*

Proof. The factorization is a consequence of Lemmas A.10 and A.11 and the small object argument [10, 10.5.16]. The second claim follows by Lemma A.7 and the colimit axiom.

We prove the last claim. Let $f\colon X \to Y$ be a map in $E \cap \mathrm{inj}\,(E \cap C)$. Since $W \subset E$ and $W \cap F = \mathrm{inj}\,(C) \subset \mathrm{inj}\,(E \cap C)$ it follows that $\mathrm{inj}\, C \subset E \cap \mathrm{inj}\,(E \cap C)$. We now prove that any map f in $E \cap \mathrm{inj}\,(E \cap C)$ is a retract of a map in $\mathrm{inj}\, C$, hence in $\mathrm{inj}\, C$. There is a factorization $X \xrightarrow{i} Z \xrightarrow{g} Y$ of f such that i is in C and g is in $W \cap F$. Lemma A.6 gives that i is also in E, hence in $E \cap C$. Since f has the right lifting property with respect to i the diagram

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

lifts. Hence f is a retract of g. ☐

Let $\mathcal{H}$ denote the heart homology. Proposition A.12 applied to the homology functor $h(X) = \bigoplus_p \mathcal{H}(X)_p$ and $n = \infty$ gives Proposition 6.1. Proposition A.12 applied to the homology functors $h_n(X) = \bigoplus_p (\mathcal{H}_{d(p)+n}(X))_p$, for all n, gives the t-model structures in Proposition 7.3.

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