Chain Rule in Functor Calculus

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Introduction

Functor calculus is a theory for studying homotopy functors by approximating them in a systematic way. The development started about ten years ago and most of the results are due to Thomas G. Goodwillie. The main applications have been in Algebraic K-theory. There is an analogy with the ordinary calculus, where (stably excisive) homotopy functors plays the role of continuous real functions, and the category of spaces plays the role of the real numbers. Given a homotopy functor F, for each space X there is a Goodwillie tower of functors from spaces over X to based spaces

$$\cdots \to P_n F \to P_{n-1} F \to \cdots \to P_1 F \to P_0 F$$

and compatible maps $p_n F : F \to P_n F$ (the maps are natural transformations). The n-th stage $P_n F$ is analogous to the n-th Taylor polynomial. The n-th layer of the tower is the "difference" between $P_n F$ and $P_{n-1} F$, which is the homotopy fiber of $P_n F \to P_{n-1} F$. In some sense this tower can converge. F maps to the (homotopy) limit of the tower. If F is ρ-analytic, then this map is an equivalence on spaces $(Y \to X)$ over X which are at least $(\rho+1)$-connected. $P_n F$ is n-excisive, which means that it takes strongly (homotopy) pushout $(n+1)$-cubes to pullback cubes. It also has the property that the connectivity of the natural map $p_n F(Y \to X)$ is about $n+1$ times the connectivity of $(Y \to X)$. This property makes $1 \to P_n$ universal among maps from the identity functor to functors associating n-excisive functors to stably excisive homotopy functors.

Goodwillie showed that the n-th layer functors are on the form

$$X \mapsto \Omega^\infty(\underline{C} \wedge X^{\wedge n})_{h\Sigma_n}$$

where $\underline{C}$ is a spectrum with Σ_n action, and $h\Sigma_n$ means the homotopy orbit space ($Y_{h\Sigma_n} = Y \wedge_{\Sigma_n} (E\Sigma_n)_+$). Σ_n is the n-th permutation group. Ω^∞ of a spectrum is its underlying infinite loop space.

In this paper we will be concerned with the first layer of the tower at a space X. This is a linear functor from spaces over X. Linear means 1-excisive and reduced at the terminal object 1_X. The derivative of a functor F at (X, x) is the coefficient spectrum at x of the first layer of F at X. Roughly one may think of the derivative at (X, x), as the difference (hofib) of F evaluated at $(X \vee_x S^n \to X)$ and $(X = X)$. The spectrum may be regarded as the limit process of taking smaller and smaller perturbations. Two spaces are viewed as being close to each other when there are highly connected maps between them.

The use of functor calculus has mainly been to compare functors. Under "convergence" assumptions (F and G are ρ-analytic functors to connected spaces) Goodwillie showed: If a natural transformation $F \to G$ induces an equivalence on the derivatives at any space, then the map between the homotopy fibers of a map is an equivalence, if the connectivity of the map is at least as large as some fixed constant $(\rho+1)$.

The main concern of this paper will be the question: *What is the the derivative of a composite functor $G\circ F$, in terms of the derivatives of G and F?* The hypothesis has been that we can reconstruct the derivative of $G\circ F$ at (X,x) from the derivative of F at (X,x) (reduced at various points of $F(X)$), the derivatives of G at $(F(X),y)$ for various points y in $F(X)$, and actions of $\Omega_y F(X)$ on them. In the cases when $F(X)$ is path connected the derivative of $G\circ F$ is conjectured to be the bar construction (homotopy balanced product) of the derivative of F at (X,x) and the derivatives of G at $F(X)$ over the loop space $\Omega F(X)$.

I have not been able to answer this question properly. The motivating first example is the identity functor. This is a functor to unbased spaces. Throughout the paper I have for technical reasons used functors to based spaces. It may well be possible that it is a more natural setting for the chain rule problem to work with unbased functors.

The second aim of this paper is to study a spectral sequence which for an arbitrary homology theory $G_*=\pi_*\circ G$ where G is a linear functor, and a map $Y\xrightarrow{f}X$ to a connected space X, is:

$$E^1_{p,q}=G_q(\mathrm{hofib}(f)\wedge(\Omega X)^{\wedge p})\Longrightarrow\pi_{p+q}(\mathrm{hofib}G(f)).$$

The d^1-differential involves an ΩX-action on $\mathrm{hofib}(f)$. My hope was that this spectral sequence could be used to study the chain rule problem. Let f be the map $F(X\vee_x S^n)\to F(X)$. Assume $F(X)$ to be path connected. The spectral sequence can now be interpreted as having as E^1-term the G-homology of the derivative of F at (X,x) smashed with a multiple of $\Omega F(X)$, and converging to the derivative of $G\circ F$ at (X,x). The problem is that we can only assume G to be a linear functor when $F(X)$ is equivalent to a point. In this case we shall show that the derivative of $G\circ F$ at (X,x), is just the smash product of the derivative of F at (X,x) and the derivative of G at $F(X)$. In the general case we can assume G to be a linear functor from spaces over $F(X)$. Such a functor gives rise to a homology theory over $F(X)$. So to be able to use this kind of a spectral sequence in the general case we have to extend it to the setting of spaces over X, for arbitrary X. This I have not been able to do.

This part is given in great point set topological detail. Sections 3 and 4 do not depend on sections 1 and 2 and may well be regarded as an independent part.

I am grateful to my advisor John Rognes. I like to thank him for all he has taught me, and for friendly guidance.

Conventions

A homotopy functor (in the wide sense) is a functor taking equivalences to equivalences. We will primarily use this term for functors from spaces to spaces, respecting weak equivalences. Sometimes it will be convenient to assume that they also respect homotopy equivalences.

We will work in the compactly generated category $\mathcal{U}$ (see Section 3). All spaces are assumed to be cofibrant (nondegenerated), that is, all inclusions of points are cofibrations. Some of the statements would be wrong otherwise.

We will shift between Moore paths (paths of various indexing lengths), and ordinary paths (paths of fixed indexing lengths) as needed. I prefer Moore paths, since they have associative path compositions, and strict identities. The two kinds of loop spaces are equivalent as H-groups. More generally the two corresponding variants of homotopy fibers are equivalent, and the equivalence respects the canonical loop space action to homotopy. In this way we avoid having to work only up to homotopies. For more on this, and my conventions see subsection 4.1.

We will use reduced suspensions and cones in the category of based spaces, which is denoted $\mathcal{T}$, and in general unreduced suspensions and cones in $\mathcal{U}$. Under our cofibration assumption reduced and unreduced suspensions are homotopy equivalent. Throughout the text $\operatorname{conn}(X \xrightarrow{f} Y)$ denotes the connectivity of a map $f : X \to Y$.

Organization of the text.

In 1.1 we will give the construction of the first stage in the Goodwillie tower. Lemmas 1 and 2 are standard and show to what extent homotopy (co) limits respect equivalences for some special diagrams. I have only writen down the proof of the first. In 1.2 we define (pre)spectra, and describe a coefficient spectrum functor. In 1.3 the derivative of a functor is defined, and we show that a linear functor has "constant" derivative.

Section 2 is the main part of the paper. We first show that in the case when $F(X)$ is equivalent to a point the derivative of the composite functor $G \circ F$ is the smash product of the derivatives of F and G. In 2.1 and 2.2 we describe actions on the derivatives. In 2.3 we discuss the question: What is $\operatorname{hofib} G(\gamma)$ in terms of G, $\operatorname{hofib}(\gamma)$, and the action of ΩX? Here γ is a map in the category of spaces over X, G is a linear functor from the category of spaces over X to the category of based spaces.

In Section 3 we give a list of properties of compactly generated spaces for use in Section 4. They are mostly trivial, but not all; a reference is [Steen].

In Section 4 we will build a spectral sequence. In 4.1 we describe Moore paths. In 4.2 we give a result of Ganea regarding a mixing of a homotopy cofiber and a homotopy fiber, in the context of Moore paths. In our case the choice of some of the maps is canonical, and gives better point-set control of the desired equivalence. This Section strongly follows [Gan]. In 4.3 we iterate the construction in 4.2 to get a tower, and use an exact couple to get a spectral sequence. In 4.4 we recognize the d^1-differential as a Hopf construction. Then we see some examples, and a related result of Stasheff, answering the question in 2.3 in the case when X is a point. In 4.5 my attempts to extend this to the setting of spaces over X are given.

Appendix 1 gives a compact presentation of some results on simplicial spaces, the bar construction and homotopy limits. It closely follows [Ho,Vo]. Appendix 2 contains a standard cell induction proof needed in 4.2.

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Contents

1 Calculus

1.1 Excisivity

In this paper equivalences means weak homotopy equivalences, in the sense that $f : X \to Y$ induces isomorphisms $f_* : \pi_q(X, x) \to \pi_q(Y, f(x))$ for all q and all $x \in X$. f is an n-equivalence if for all $x \in X$, f_* is an isomorphism for $q < n$, and an epimorphism for $q = n$.

Given a square
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_{12} \end{tikzcd}$$
there are canonical maps $Y \to \mathrm{holim}(Y_2 \to Y_{12} \leftarrow Y_1)$ and $\mathrm{hocolim}(Y_2 \leftarrow Y \to Y_1) \to Y_{12}$. The square is called cartesian if the first map is an equivalence, and cocartesian if the second map is an equivalence. If the first (second) map is an n-equivalence the square is called n-(co)cartesian. Definitions of homotopy (co)limits in these particular cases are given on page 34.

$\mathcal{U}_X$ is the category of spaces over X (in $\mathcal{U}$). The objects are maps $f : D(f) \to X$. $D(f)$ is the domain of f. Morphisms $\gamma : f \to g$ are maps $\gamma : D(f) \to D(g)$ so that $g \circ \gamma = f$.

Equivalences of maps and (co)cartesianness of squares in $\mathcal{U}_X$ is defined by regarding them as diagrams in $\mathcal{U}$ by the forgetful functor $\mathcal{U}_X \to \mathcal{U}$. The cone $C_X f$ is $\mathrm{hocolim}(f \to 1_X)$ and the suspension $S_X f$ is $\mathrm{hocolim}(1_X \leftarrow f \to 1_X)$. Note $C_X f \simeq 1_X$ in $\mathcal{U}_X$. A functor $F : \mathcal{U}_X \to \mathcal{T}$ is stably excisive if for some c and κ it takes every cocartesian square

$$\begin{tikzcd} f \arrow[r] \arrow[d] & f_1 \arrow[d] \\ f_2 \arrow[r] & f_{12} \end{tikzcd}$$

with maps $f \to f_i$, k_i-connected to a $(k_1 + k_2 - c)$-cartesian square when $k_1, k_2 \geq \kappa$. We then say that F satisfies $E(c, \kappa)$. A functor is excisive if it takes cocartesian squares to cartesian squares.

A functor F is reduced if $F(1_X)$ is contractible. A functor which is both excisive and reduced is called linear. An example of a linear functor is $Q = \Omega^\infty S^\infty : \mathcal{T} \to \mathcal{T}$, the functor defined by $Q(X) = \mathrm{hocolim}(\Omega^\ell S^\ell X)$. Here the maps are induced by the unit $1 \to \Omega S$ in the loop-suspension adjunction (here S is reduced suspension).

Definition. $\mathcal{F}_X$ is the functor category consisting of all stably excisive homotopy functors, and $\mathcal{E}_X$ is the functor category consisting of all excisive homotopy functors. A map $\Gamma : F \to G$ is an equivalence if $\Gamma(f)$ is an equivalence for each $f \in \mathcal{U}_X$. Maps of functors are also called natural transformations.

Remark. $\mathcal{F}_X(F,G) \subset \Pi_{f\in\mathcal{U}_X}\mathcal{U}(F(f),G(f))$ is in general not a set. We like to have categories with small morphism classes. To obtain this we will restrict $\mathcal{F}_X$ to the category of stably excisive homotopy functors $U \to \mathcal{T}$ for small full subcategories U of $\mathcal{U}_X$. The functors are restricted to U, and the morphism sets are $\Pi_{f\in U}\mathcal{U}(F(f),G(f)) \cap \mathcal{F}_X(F,G)$. They are topologized by the subspace topology. In practice we will choose U so as to contain the spaces we need. The result will be independent of which choice of U we make as soon as it is large enough.

Construction. We will now construct the first layer in the Goodwillie tower. It will be a homotopy functor $P : \mathcal{F}_X \to \mathcal{E}_X$.

There is a functor $T : \mathcal{F}_X \to \mathcal{F}_X$ taking an $E(c,\kappa)$ functor to an $E(c-1,\kappa-1)$ functor. It is defined by $TF(f) = \operatorname{holim}(F(C_X f) \to F(S_X f) \leftarrow F(C_X f))$. Here $S_X f$ is regarded as the union of two cones along f. There is a canonical natural transformation $t : 1_{\mathcal{F}_X} \to T$. The assertion that TF satisfies $E(c-1,\kappa-1)$ follows by commutativity of homotopy limits, and the lemmas below.

In the following diagram a and c are equivalences, and b is n-connected.

$$\begin{array}{cccccc} \mathrm{I} & A & \longrightarrow & B & \longleftarrow & C \\ & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ \mathrm{II} & A' & \longrightarrow & B' & \longleftarrow & C' \end{array}$$

Lemma 1 $\operatorname{holim}(\mathrm{I}) \to \operatorname{holim}(\mathrm{II})$ *is* $(n-1)$*-connected.*

This is equivalent to that $\operatorname{hofib}(\operatorname{holim}(\mathrm{I}) \to \operatorname{holim}(\mathrm{II}))$ is $(n-2)$-connected for all basepoints in $\operatorname{holim}(\mathrm{II})$, and that it is a π_0-epimorphism when $n \geq 1$. I will show this in two parts.

Part 1. We will here show that the homotopy fiber of $\operatorname{holim}(\mathrm{I}) \to \operatorname{holim}(\mathrm{II})$ at an arbitrary point (a',ω,c') in $\operatorname{holim}(\mathrm{II})$ is $(n-2)$-connected.

$$\operatorname{holim}\begin{pmatrix} \operatorname{holim}(\mathrm{I}) \\ \downarrow \\ \operatorname{holim}(\mathrm{II}) \\ \uparrow \\ (a',\omega,c') \end{pmatrix} \longrightarrow \operatorname{holim}\begin{pmatrix} \operatorname{holim}(\mathrm{I}) \\ \downarrow \\ \operatorname{holim}(\mathrm{II}) \\ \uparrow \\ \operatorname{holim}(a' \to \mathrm{I} \leftarrow c') \end{pmatrix}$$

In the second space the upward map is given by inclusions of the points, and the map $\mathrm{I} \xrightarrow{\omega} B'$. The first space is the homotopy fiber at (a',ω,c'). The map is induced by $(a',\omega,c') \mapsto (a',1_{\mathrm{I}},c')$. $\operatorname{holim}(a' \to \mathrm{I} \leftarrow c')$ is the subspace of $\mathcal{U}(\mathrm{I},\mathrm{I})$ consisting of all maps taking 0 to 0 and 1 to 1. It is contractible. By homotopy invariance of homotopy limits, the horizontal map is a homotopy equivalence. By commutativity of homotopy limits, the last space is homotopy equivalent to

$$\operatorname{holim}(\operatorname{hofib}_{a'}(A \to A') \to \operatorname{holim}(B \to B' \xleftarrow{\omega} \mathrm{I}) \leftarrow \operatorname{hofib}_{c'}(C \to C'))$$

Since the homotopy fiber of an equivalence is weakly contractible

$$\mathrm{holim}(* \to \mathrm{holim}(B \to B' \xleftarrow{\omega} \mathrm{I}) \leftarrow *)$$

maps by an equivalence into the preceeding space. This homotopy limit is homotopy equivalent to $\Omega_y \mathrm{holim}(B \to B' \xleftarrow{\omega} \mathrm{I})$ for some basepoint y.

$$\mathrm{hofib}_{\omega(0)}(b) = \mathrm{holim}(B \to B' \xleftarrow{\omega(0)} *) \to \mathrm{holim}(B \to B' \xleftarrow{\omega} \mathrm{I})$$

The map is induced by $* \xrightarrow{\mathrm{to}\,0} \mathrm{I}$, hence is a homotopy equivalence. $\mathrm{hofib}(b)$ is $(n-1)$-connected for every basepoint and Ω lowers connectivities by one. Hence $\mathrm{hofib}(\mathrm{holim}(\mathrm{I}) \to \mathrm{holim}(\mathrm{II}))$ is $(n-2)$-connected.

Part 2. We will here show that for any point in holim(II), there is a path in holim(II) from this to a point in the image of holim(I).

Take an arbitrary point (a', ω, c') in holim(II). By assumption on the maps a and c, there are points $\hat{a}$ in A, and $\hat{c}$ in C so that a' and $a(\hat{a})$ are in the same path component, and c' and $c(\hat{c})$ are in the same path component. Paths between these points determine an element $(a(\hat{a}), \sigma, c(\hat{c}))$ in holim(II), and a path from it to (a', ω, c'). The images of $\hat{a}$ and $\hat{c}$ in B must be in the same path component since $\pi_0(b)$ is injective. Hence there is a path τ from the image of $\hat{a}$ to the image of $\hat{c}$. $b(\tau)$ is a path from the image of $a(\hat{a})$ to the image of $c(\hat{c})$. $\sigma * (b(\tau))^{-1}$ is a loop at the image of $a(\hat{a})$, and $a(\hat{a})$ is b of the image of $\hat{a}$ in B. Since $\pi_1(b)$ is epic at all points in B, there is a loop ζ at the image of $\hat{a}$ so that $b(\zeta)$ is homotopic to $\sigma * (b(\tau))^{-1}$. Hence σ is homotopic (relative to start and end points) to $b(\zeta) * b(\tau)$. So (a', ω, c') is in the same path component as $b(\hat{a}, \zeta * \tau, \hat{c})$. □

Lemma 2 *S_X increases connectivity by at least one.* □

Definition.

$$PF = \mathrm{hocolim}(F \xrightarrow{t(F)} TF \xrightarrow{t(TF)} T^2F \to \cdots)$$

T^nF satisfies $E(c-n, \kappa-n)$. By right cofinality, that homotopy colimits takes n-equivalences to n-equivalences, and finally that in this case of filtered homotopy colimit and finite homotopy limit they naturally homotopy commute, it follows that PF satisfies $E(c', \kappa')$, for arbitrary c' and κ', hence is excisive. There is a natural transformation $p : 1_{\mathcal{F}_X} \to P$. If F is a reduced based functor, there is a natural equivalence $\Omega F(S_X f) \hookrightarrow TF(f)$. TF is again a reduced based functor, so we have equivalences $\Omega^n F(S_X^n f) \hookrightarrow T^nF(f)$. Informally one may write $PF(f) \simeq \mathrm{hocolim}\, \Omega^n F(S_X^n f)$. There are not in general canonical maps between the spaces in this homotopy colimit. One map is determined by a choice of inverses of the (homotopy) equivalences.

The reduction functor Red : $\mathcal{F}_X \to \mathcal{F}_X$ is given by $F \mapsto \widetilde{F}$ where $\widetilde{F}(f) = \mathrm{hofib}(F(f) \to F(1_X))$. Note that 1_X is a terminal object in $\mathcal{U}_X$. The composite functor Red $\circ\, P$ is denoted D. $\widetilde{F}(1_X)$ is contractible. Since a homotopy limit of cartesian squares is cartesian, D is a functor to the category of linear functors. There is a natural transformation Red $\to 1_{\mathcal{F}_X}$. Let $\widetilde{\mathcal{F}_X}$ denote the subcategory of all reduced functors, and let $\widetilde{\mathcal{E}_X}$ denote the subcategory of all linear functors.

The composite functor $\mathcal{F} \hookrightarrow \mathcal{F}_X \xrightarrow{P} \mathcal{E}_X$ is denoted P_X. The further composition with the reduction functor is denoted D_X. $D_X F$ and $P\widetilde{F}$ are naturally equivalent; this follows from the next lemma.

A map $\Gamma : F \to G$ is said to be cartesian if for all maps $\theta : f \to g$ the following diagram is cartesian.
$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] & G(f) \arrow[d] \\ F(g) \arrow[r] & G(g) \end{tikzcd}$$

A map Γ between functors $\mathcal{F}_X \to \mathcal{F}_X$ is said to be cartesian if $\Gamma(F)$ is cartesian for all $F \in \mathcal{F}_X$. The same definitions apply with cartesian replaced by cocartesian.

Lemma 3 *The map $P \circ \mathrm{Red} \to P$ is cartesian.*

For all $F \in \mathcal{F}_X$, $\widetilde{F} \to F$ is cartesian. This is seen by taking homotopy fibers of the diagram
$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] & F(1_X) \arrow[d] \\ F(g) \arrow[r] & F(1_X). \end{tikzcd}$$
By commutativity of homotopy limits, all the $T^i\widetilde{F} \to T^iF$ are also cartesian maps. Since finite homotopy limits naturally homotopy commute with the homotopy colimit in the construction of P it follows that $P\widetilde{F} \to PF$ is also cartesian. $\square$

Applied to the case
$$\begin{tikzcd} P\widetilde{F}(f) \arrow[r] \arrow[d] & PF(f) \arrow[d] \\ P\widetilde{F}(1_X) \arrow[r] & PF(1_X) \end{tikzcd}$$
we get a sequence of equivalences in the category $\widetilde{\mathcal{E}_X}$
$$P\widetilde{F} \leftarrow \widetilde{P\widetilde{F}} \rightarrow \widetilde{PF} = DF.$$

1.2 Spectra

We will use omega (pre)spectra. There are more rigid categories of spectra, but in our applications the omega spectra occur most directly. A prespectrum $\underline{A}$ is a sequence of based spaces $\{A_i\}_{i\geq 0}$, and structure maps $\alpha_i : A_i \to \Omega A_{i+1}$. A map of prespectra $\theta : \underline{A} \to \underline{B}$ is a sequence of maps $\theta_i : A_i \to B_i$ so that $\beta_i \circ \theta_i = \Omega\theta_{i+1} \circ \alpha_i$.

A prespectrum $\underline{A}$ is called a spectrum if each α_i is an equivalence. θ is an equivalence of spectra if each θ_i is an equivalence. Denote the category of prespectra by $p\mathcal{S}$, and the category of spectra by $\mathcal{S}$.

For each point $x \in X$, define the element s_n of $\mathcal{U}_X$ as follows. $s_n : X \vee_x S^n \to X$ is the identity on X, and the collapse map to x on S^n. There is a canonical homotopy equivalence $S_X s_n \to s_{n+1}$.

Prespectrum functor. For each point x in X, we define a homotopy functor $\mathbb{K}_x : \widetilde{\mathcal{F}}_X \to p\mathcal{S}$.

The homotopy fibers of the diagram

$$\begin{tikzcd}
F(s_n) \arrow[r, equal] \arrow[d] & F(s_n) \arrow[r] \arrow[d] & F(C_X s_n) \arrow[r, equal] \arrow[d] & F(C_X s_n) \arrow[d] & F(1_X) \arrow[l, "\simeq"'] \arrow[d] & * \arrow[l, "\simeq"'] \arrow[d] \\
* & F(C_X s_n) \arrow[l, "\simeq"'] \arrow[r] & F(S_X s_n) \arrow[r, "\simeq"] & F(s_{n+1}) \arrow[r, equal] & F(s_{n+1}) \arrow[r, equal] & F(s_{n+1})
\end{tikzcd}$$

give the natural maps

$$F(s_n) \xleftarrow{\simeq} \#_n \longrightarrow \flat_n \xleftarrow{\simeq} \Omega F(s_{n+1})$$

When F is excisive the middle map is also an equivalence. (It should be clear what the homotopy fibers $\#_n$ and $\flat_n$ are.)

$\mathbb{K}_x(F) = \{F_n, f_n\}$ is defined by

$$F_n = \operatorname{holim}(F(s_n) \leftarrow \#_n \to \flat_n \leftarrow \Omega F(s_{n+1}) \leftarrow \Omega\#_{n+1} \to \dots).$$

The structure map $f_n : F_n \to \Omega F_{n+1}$ is induced by the restriction of diagrams

$$\begin{aligned}
&\operatorname{holim}(F(s_n) \leftarrow \#_n \to \flat_n \leftarrow \Omega F(s_{n+1}) \leftarrow \Omega\#_{n+1} \to \dots) \\
&\qquad \longrightarrow \operatorname{holim}(\Omega F(s_{n+1}) \leftarrow \Omega\#_{n+1} \to \dots).
\end{aligned}$$

This map is obviously natural. The last space maps by a natural homeomorphism to ΩF_{n+1}. It is an equivalence if F is excisive. Thus the functor restricted to $\widetilde{\mathcal{E}_X}$ gives a functor to spectra.

The inclusion of $F(s_n)$ into the diagram in the definition of F_n induces a map $F_n \to F(s_n)$. This map is an equivalence. This is seen by comparing the diagram with one where the equivalences are reversed. For there $F(s_n)$ is in an initial position.

Spectrification functor. There is a homotopy functor $\mathbb{L} : p\mathcal{S} \to \mathcal{S}$. Suppose given a pre-spectrum $\underline{A}$. The spaces and structural maps of $\mathbb{L}\underline{A}$ are defined by

$$\begin{tikzcd}
\mathbb{L}(\underline{A})_n = \operatorname{hocolim}(& A_n \arrow[r] \arrow[d] & \Omega A_{n+1} \arrow[r] \arrow[d] & \Omega^2 A_{n+2} \arrow[r] \arrow[d] & \cdots) \\
\operatorname{hocolim}(& \Omega A_{n+1} \arrow[r] & \Omega^2 A_{n+2} \arrow[r] & \Omega^3 A_{n+3} \arrow[r] & \cdots)
\end{tikzcd}$$

The last space maps by a natural homotopy equivalence to $\Omega\mathbb{L}(\underline{A})_{n+1}$. These maps are equivalences. For the map $\mathbb{L}(\Omega\underline{A})_{n+1} \to \mathbb{L}(\underline{A})_n$ is an equivalence by cofinality, and the further composition to $\mathbb{L}(\Omega\underline{A})_{n+1}$ is homotopic to the identity.

There is a natural transformation $1_{p\mathcal{S}} \to \mathbb{L}$. When $\underline{A}$ is a spectrum the map $\underline{A} \to \mathbb{L}(\underline{A})$ is an equivalence, since each map in the homotopy colimit is an equivalence. So spectrification of a spectrum gives an equivalent spectrum.

A map of prespectra is defined to be an equivalence if the spectrification is an equivalence.

1.3 The derivative of a functor

Given homotopy functors $F, G : \mathcal{U}_X \to \mathcal{T}$. A map $\gamma : F \to G$ satisfies $O(c, \kappa)$ if $\gamma(f)$ is $(2\,\mathrm{conn}(f) - c)$-connected whenever $\mathrm{conn}(f) \geq \kappa$. Replacing 2 by $n + 1$, we say that the functors agree to n-th order.

If $\gamma : F \to G$ satisfies $O(c, \kappa)$ then $T\gamma : TF \to TG$ satisfies $O(c-1, \kappa-1)$, hence $P\gamma : PF \to PG$ is an equivalence. The argument is the same as for $E(c, \kappa)$ above. $E(c, \kappa)$ for F implies that $t(F) : F \to TF$ satisfies $O(c, \kappa)$, whence $p(F) : F \to PF$ satisfies $O(c, \kappa)$. $\tilde{F}$ also satisfies $E(c, \kappa)$, so $p(\tilde{F})$ satisfies $O(c, \kappa)$. Application of Lemma 3 shows that $\tilde{F} \to DF$ satisfies $O(c, \kappa)$.

The coefficient spectrum of a linear functor $L : \mathcal{U}_X \to \mathcal{T}$ at $x \in X$ is the spectrum $\mathbb{K}_x(L)$. The i-th space is equivalent to $L(X \vee_x S^i \to X)$. For a functor F in $\mathcal{F}_X$, we define the derivative of F at (X, x) to be $\mathbb{K}_x(P\tilde{F})$. This is denoted $\underline{\partial}_x F(X)$. This is equivalent (as a spectrum) to $\mathbb{K}_x(D_X F)$, which Goodwillie defines to be the derivative. Our definition will be more convenient in the next section. (This can be compared with taking the coefficients of the linear part of the Taylor series of $f(x) - f(a)$ or of $f(x)$ at a.)

Proposition 1 *If F, G and $G \circ F$ are stably excisive functors from $\mathcal{U}$ to $\mathcal{T}$, and F takes n-equivalences to $(n - c)$-equivalences for n large enough, then there is a sequence of equivalences connecting $\underline{\partial}_x(G \circ F)(X)$ and $\underline{\partial}_x(P\tilde{G} \circ F)(X)$*

F determines a functor $\mathcal{U}_X \to \mathcal{U}_{F(X)}$. G is considered as a functor $\mathcal{U}_{F(X)} \to \mathcal{T}$ by composing it with the forgetful functor $\mathcal{U}_{F(X)} \to \mathcal{U}$. Let $\tilde{G}$ denote G reduced at $F(X)$. The map $p(\tilde{G}) : \tilde{G} \to P\tilde{G}$ agrees to first order. By assumption on F, $\tilde{G} \circ F \to P(\tilde{G}) \circ F$ also agrees to first order. So $P(\widetilde{G \circ F}) \to P(P(\tilde{G}) \circ F)$ is an equivalence, $\tilde{G} \circ F = \widetilde{G \circ F}$. By use of the natural equivalence $P(\widetilde{P\tilde{G} \circ F}) \to P(P\tilde{G} \circ F)$ we have the following sequence of spectrum equivalences.

$$\underline{\partial}_x(G \circ F)(X) \to \mathbb{K}_x(P(P\tilde{G} \circ F)) \leftarrow \underline{\partial}_x(P\tilde{G} \circ F)(X) \quad \square$$

So when we study the derivative of a composite functor $G \circ F$, we can assume that G is a linear functor $\mathcal{U}_{F(X)} \to \mathcal{T}$.

When $F \in \mathcal{E}$, so is $\widetilde{F} \in \mathcal{E}_X$, hence $\widetilde{F} \to P\widetilde{F}$ is an equivalence. Applying F to the cocartesian diagram

$$\begin{tikzcd} X \vee_x S^n \arrow[r] \arrow[d] & S^n \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$

gives the sequence of equivalences

$$P\widetilde{F}(s_n) \leftarrow \widetilde{F}(s_n) \rightarrow \operatorname{hofib}(F(S^n) \rightarrow F(*)) \rightarrow P\widetilde{F}(S^n)$$

In the last space F is reduced at a point. The structure maps are respected, this is seen by writing down the diagrams. This shows that in the case of an excisive functor, the derivative is independent of where we derive.

2 Composition of functors

We shall here discuss what a chain rule in functor calculus can be.

Let F and G be stably excisive homotopy functors. To be sure that also the composite functor $G \circ F$ is a stably excisive functor, we will restrict the discussion to a subcategory of $\mathcal{F}_X$. A functor satisfies $D(c, \kappa)$ if it takes n-equivalences to $(n-c)$-equivalences for some constant c and all $n \geq \kappa$. We will call a functor differentiable if it is continuous, and satisfies $E(c, \kappa)$ and $D(c', \kappa')$ for some constants c, κ, c' and κ'. The continuity requirement will be necessary when we will try to define actions on the derivatives. We will show that the category of differentiable functors from $\mathcal{U}$ to $\mathcal{T}$ is closed under compositions, where we use the forgetful functor $\mathcal{T} \to \mathcal{U}$. To prove this we will make use of the Blakers-Massey triad connectivity theorem, and its dual. For proofs of these theorems and their generalizations, see [Good2]. The following diagram is for the statements below.

$$\begin{array}{ccc} X & \xrightarrow{k_1} & X_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_2} \\ X_2 & \xrightarrow{l_1} & X_{12} \end{array}$$

Theorem 1 *(Blakers-Massey triad connectivity theorem.) If the square is cocartesian then the square is also* $(\mathrm{conn}(k_1) + \mathrm{conn}(k_2) - 1)$*-cartesian.* $\square$

Theorem 2 *(Dual Blakers-Massey triad connectivity theorem.) If the square is cartesian then the square is also* $(\mathrm{conn}(l_1) + \mathrm{conn}(l_2) + 1)$*-cocartesian.* $\square$

The Blakers-Massey triad connectivity theorem is equivalent to the statement that the identity functor satisfies $E(1, \kappa)$ for all κ.

Proposition 2 *If F and G are differentiable functors then the composite functor $G \circ F$ is also a differentiable functor.*

Let F satisfy $E(c, \kappa)$ and $D(c', \kappa')$, and let G satisfy $E(d, \lambda)$ and $D(d', \lambda')$. For brevity we denote the maps by their connectivities. In this proof the index i is 1 or 2.

Assume the square above to be cocartesian. By comparing homotopy cofibers we have that $l_i \geq 0$ if $k_i \geq 0$. By the Blakers-Massey theorem the square is $(k_1 + k_2 - 1)$-cartesian. By comparing homotopy fibers, and by the 0-connectivities of l_i we get $l_i \geq k_i$ whenever $k_1, k_2 \geq 1$. Note that we have to compare homotopy fibers taken in each path component. Apply F to the square diagram. The connectivities of the maps are $F(k_i) \geq k_i - c'$ whenever $k_i \geq \kappa'$ and $F(l_i) \geq k_i - c'$ whenever $k_i \geq \kappa'$ and $k_1, k_2 \geq 1$. We will compare this square with a cocartesian square. We then apply G, and compare cartesianness of the resulting square diagrams. In the following diagram the bullets are used for brevity.

$$\begin{tikzcd}
& \bullet \arrow[rr, equal] \arrow[dd] & & \bullet \arrow[rr, equal] \arrow[dd] & & F(X_1) \arrow[dd, "F(l_2)"] \\
\bullet \arrow[ur] \arrow[rr, equal, crossing over] \arrow[dd] & & M \arrow[ur, "m_1"] & & F(X) \arrow[ll, "m_0"'] \arrow[ur, "F(k_1)"'] & \\
& N \arrow[rr, "n"] & & \bullet \arrow[rr, equal] & & F(X_{12}) \\
\bullet \arrow[ur] \arrow[rr, equal] & & \bullet \arrow[ur] \arrow[rr, equal] \arrow[from=uu, crossing over, "m_2"] & & F(X_2) \arrow[ur, "F(l_1)"'] \arrow[from=uu, crossing over, "F(k_2)"] &
\end{tikzcd}$$

In the diagram

$$M = \operatorname{holim}(F(X_1) \to F(X_{12}) \leftarrow F(X_2)\,) \text{ and}$$

$$N = \operatorname{hocolim}(F(X_1) \leftarrow M \to F(X_2)\,).$$

The lower bounds of the connectivities of the maps m_0, m_1 and m_2 are

$$m_0 \geq k_1 + k_2 - c \text{ whenever } k_1, k_2 \geq \kappa$$

$$m_i \geq \min(k_i - c', m_0 - 1) \text{ whenever } k_1, k_2 \geq \kappa \text{ and } k_i \geq \kappa'.$$

The middle square is cartesian. Hence by the dual Blakers-Massey theorem it is $(F(l_1) + F(l_2) + 1)$-cocartesian.

$$n \geq k_1 + k_2 + 1 - 2c' \text{ whenever } k_1, k_2 \geq \max(\kappa', 1).$$

The square on the left is cocartesian. Applying G to the diagram we get

$$\begin{tikzcd}
G(M) \arrow[r, equal] \arrow[d, "p"] & G(M) \arrow[d] & GF(X) \arrow[l, "G(m_0)"'] \arrow[d, "r"'] \\
\operatorname{holim}(GF(X_1) \to G(N) \leftarrow GF(X_2)) \arrow[r, "q"] & \bullet \arrow[r, equal] & \operatorname{holim}(GF(X_1) \to GF(X_{12}) \leftarrow GF(X_2)).
\end{tikzcd}$$

We have the following connectivity estimates of the maps

$$p \geq m_1 + m_2 - d \text{ whenever } m_1, m_2 \geq \lambda$$

and the map q is $(G(n) - 1)$-connected by Lemma 1.

Putting this together shows that

$$r \geq \min(p, G(n) - 1, G(m_0)\,).$$

All these maps are ($k_1 + k_2-$ some constant)-connected whenever k_1, k_2 are large enough. Hence $G \circ F$ is stably excisive. $G \circ F$ satisfies $D(c' +$

$d', \max(\kappa', c'+\lambda'))$ and is continuous, whence $G\circ F$ is a differentiable functor. For completeness I will write down the details in the connectivity estimate of r.

$$r \geq k_1 + k_2 - \max\{c + d', 2c' + d', \Sigma_{i=1}^{2}\max(c', c+1-k_i) + d\}$$

Whenever k_1, k_2 satisfies the following equations

$$\begin{array}{rr} k_1, k_2 \geq \max(\kappa', 1, \kappa), & k_1 - \max(c', c+1-k_2) \geq \lambda \\ k_1 + k_2 \geq \lambda' + \max(c, 2c'-1), & k_2 - \max(c', c+1-k_1) \geq \lambda. \end{array}$$

In particular this is satisfied if

$$k_1, k_2 \geq \max(\kappa', 1, \kappa, \lambda + c', \lambda' + 2c' - 2, \lambda' + c - 1).$$

□

It is easy to see that if F is a differentiable functor, then $\mathrm{Red}F$ and PF are also differentiable functors.

$s_{n,x} : X \vee_x S^n \to X$ is the n-sphere perturbation of 1_X at x. $F(s_n) : F(X \vee_x S^n) \to F(X)$ is a perturbation of $1_{F(X)}$, but the perturbation is spread out on $F(X)$. The space $F(X \vee_x S^n) \to F(X)$ is retractive. A right inverse of $F(s_n)$ is given by F applied to the map $X \overset{i}{\hookrightarrow} X \vee_x S^n$. The map $F(i)(F(X)) \xrightarrow{F(s_n)|} F(X)$ is a homeomorphism over $F(X)$.

Together this gives a perturbation $G \circ F(s_n) : G \circ F(X \vee_x S^n) \to G \circ F(X)$. It seems reasonable to expect that one can piece information of $\underline{\partial}_x F(X)$ and the derivative(s) of G at $F(X)$ together to say something about $\underline{\partial}_x(G \circ F)(X)$. The basepoint of $F(X)$ is forgotten in $(G \circ F)(X)$. So there is no reason that the derivative at $F(X)$ and its basepoint, should contain more information than the derivatives at other points of $F(X)$. Since the perturbation $F(s_n)$ of $1_{F(X)}$ is not localized to be over points, it is most reasonable that we need information of the derivatives at different points of $F(X)$ and how they are related to each other. At least we must know the derivatives at each path component of $F(X)$.

Most naively one might expect that $\underline{\partial}_x G\circ F(X)$ is just the smash product of $\underline{\partial}_x F(X)$ and $\underline{\partial}_y G(F(X))$. This is the case when $F(X) \simeq *$. The location of basepoints in $F(X)$ is then expected to be of no importance. I give a sketchy proof of this.

Here and in the rest of this section, we let $\sim$ denote a chain of equivalences. We will not denote forgetful functors like $\mathcal{T} \to \mathcal{U}$ and $\mathcal{U}_X \to \mathcal{U}_*$. $s'_n : F(X) \vee_y S^n \to F(X)$ are the n-sphere perturbation elements in $\mathcal{U}_{F(X)}$. A linear functor $L : \mathcal{U} \to \mathcal{T}$ satisfies the limit axiom if for all CW-complexes Y, the map $\mathrm{hocolim}_\alpha L(Y_\alpha) \to L(Y)$ is an equivalence, where the homotopy colimit is over all finite subcomplexes.

Proposition 3 *Let F and G be differentiable functors, and assume that P_*G satisfies the limit axiom. If $F(X) \simeq *$ then $\underline{\partial}_x G \circ F(X)$ is equivalent to $\underline{\partial}_x F(X) \wedge \underline{\partial}_y G(F(X))$.*

Here we will use subscript on the functor P to denote at what category it is a functor, $P_X : \mathcal{F}_X \to \mathcal{E}_X$.

$$\underline{\partial_x} G \circ F(X)_n \to P_X(\widetilde{G \circ F})(s_n) = P_X(\widetilde{G} \circ F)(s_n) \to P_X(P_{F(X)}\widetilde{G} \circ F)(s_n)$$

The maps are all equivalences. The homotopy equivalence $F(X) \to *$ induces a natural equivalence $P_{F(X)}\widetilde{G} \to P_*\widetilde{G}$. This is seen by writing down the diagrams constructing P in both cases (cf.p. 22). The maps between the spaces in the two diagrams are equivalences since G is a homotopy functor.

$$\to P_X(P_*\widetilde{G} \circ F)(s_n) \sim \operatorname{hocolim} \Omega^\ell (P_*\widetilde{G} \circ F)(s_{n+\ell})$$

In the last step the homotopy colimit in the definition of P_X is replaced with an equivalent diagram (cf.p. 9).

By [Good1] a linear functor $L : \mathcal{T} \to \mathcal{T}$ is given by its coefficient spectrum, $L(Y) \sim \operatorname{hocolim} \Omega^\ell(\underline{\partial} L(*)_\ell \wedge Y)$, at least when Y has the type of a finite CW-complex. When L satisfies the limit axiom this holds in general, since every space is equivalent to a CW-complex. With these assumptions we get

$$\sim \operatorname{hocolim} \Omega^\ell \operatorname{hocolim} \Omega^k (\underline{\partial} G(*)_k \wedge F(s_{n+\ell})).$$

Since $\widetilde{F} \to P_X\widetilde{F}$ agrees to first order in $\mathcal{F}_X$ it follows that in the sequence of maps

$$F(s_n) \leftarrow \widetilde{F}(s_n) \to P_X\widetilde{F}(s_n) \leftarrow \underline{\partial_x} F(X)_n$$

the middle map is $(2n-c)$-connected for some c, and n large enough. The two other maps are equivalences.

$$\sim \operatorname{hocolim} \Omega^\ell \operatorname{hocolim} \Omega^k (\underline{\partial} G(*)_k \wedge \underline{\partial_x} F(X)_{n+\ell})$$

By use of the sequence of equivalences

$$\underline{\partial_y} G(F(X))_n \to P_{F(X)}\widetilde{G}(s'_n) \xrightarrow{\simeq} P_*\widetilde{G}(S^n) \leftarrow \underline{\partial} G(*)_n$$

we have

$$\sim \operatorname{hocolim} \Omega^\ell \operatorname{hocolim} \Omega^k (\underline{\partial_y} G(F(X))_k \wedge \underline{\partial_x} F(X)_{n+\ell}).$$

Since Ω^k is a finite homotopy limit and the homotopy colimit is directed, the map of functors; $\operatorname{hocolim}_k \Omega^\ell \to \Omega^\ell \operatorname{hocolim}_k$ is an equivalence.

$$\sim \operatorname{hocolim} \Omega^{k+\ell} (\underline{\partial_y} G(F(X))_k \wedge \underline{\partial_x} F(X)_{n+\ell})$$

$$\sim \operatorname{hocolim} \Omega^j (\underline{\partial_y} G(F(X)) \wedge \underline{\partial_x} F(X))_{j+n}$$

So $(\underline{\partial_x} G \circ F(X))_n \sim (\mathbb{L}(\underline{\partial_y} G(F(X)) \wedge \underline{\partial_x} F(X))\,)_n$. Following the structural maps through the steps, one can see that they are respected up to homotopy. □

It does not hold generally that the derivative of $G \circ F$ at X is the smash product of the derivative of F at X and the derivative of G at $F(X)$. This is already seen with $F = G = \mathrm{id}$. The identity functor is not a based functor. When applied on $s_{n,x}$ we choose x as the basepoint.

$$\mathrm{hofib}_x(X \vee_x S^n \to X) = PX \cup_{\Omega X} \Omega X \times S^n \xrightarrow{\simeq} \Omega X_+ \wedge S^n.$$

We get the suspension spectrum $\Sigma^\infty \Omega X_+$. The derivative is the omega spectrum of the underlying (infinite loop) space $Q(\Omega X_+) = \Omega^\infty \Sigma^\infty(\Omega X_+)$. The smash product of these suspension spectra is

$$\Sigma^\infty \Omega X_+ \wedge \Sigma^\infty \Omega X_+ = \Sigma^\infty(\Omega X_+ \wedge \Omega X_+) = \Sigma^\infty(\Omega X \times \Omega X)_+.$$

This equals $\Sigma^\infty \Omega X_+$ typically only when ΩX is contractible.

In a chain rule, inserting the derivative of the identity must make it an identity formula on the other factor. More precisely, if the derivative of $G \circ F$ at (X, x) is given in terms of the derivatives of G and F this must be done in such a way that if F is the identity functor, $\underline{\partial}_x F(X) \simeq \Sigma^\infty \Omega X_+$ must "divide out", and if G is the identity functor, $\underline{\partial}_y G(F(X)) \simeq \Sigma^\infty \Omega F(X)_+$ must "divide out" (here $\simeq$ means equivalences of prespectra). In the case of loop composition action on the derivative of the identity functor, an extension of the smash product formula satisfying this is the smash product over $\Omega F(X)$. In the example id$\circ$id the calculation is the following: We will here use Moore loops (cf. Section 4.1). With the loop composition action of ΩX on ΩX, we have the balanced product $\Omega X \otimes_{\Omega X} \Omega X \cong \Omega X$. In the case when the actions are left and right loop compositions, it is defined by $\Omega X \times \Omega X / \sim$ where the equivalence relation $\sim$ is generated by $(\omega, \sigma * \omega') \sim (\omega * \sigma, \omega')$, where ω, ω' and σ are all loops. ($\Omega X \otimes_{\Omega X} \Omega X \to \Omega X$ is given by $[\omega, \sigma] \mapsto \omega * \sigma$; this is well defined, and the inverse map is $\omega \mapsto [1, \omega]$).

The homotopy invariant version of the balanced product is the bar construction. We may state our guess as

$$\underline{\partial}_x G \circ F(X) \simeq B(\underline{\partial}_y G(F(X)), \Omega F(X), \underline{\partial}_x F(X)).$$

This formulation does not take care of what derivative(s) of G to use. We use the derivative of G at the basepoint of $F(X)$. This should only be taken seriously in the case when $F(X)$ is path connected. The bar construction on the space level is described in Appendix 1.

From the case of the identity functor we may expect nontrivial actions of $\Omega F(X)$ on both $\underline{\partial}_y G(F(X))$ and $\underline{\partial}_x F(X)$. We will now try to give $\Omega F(X)$-actions on these spectra. At the identity functor they shall be the loop composition actions.

2.1 $\Omega F(X)$-action on $\underline{\partial}_x F(X)$

First we consider a categorical way to do this.

A functor $F : \mathcal{U} \to \mathcal{U}$ gives rise to a map $F : \mathcal{U}(A, B) \to \mathcal{U}(F(A), F(B))$. If this map is continuous for all objects A and B in $\mathcal{U}$, then F is called a continuous functor. We now assume F to be a continuous functor. $\mathcal{U}(A, A)$ is a topological monoid under composition. $F : \mathcal{U}(A, A) \to \mathcal{U}(F(A), F(A))$ is a monoid map. In $\mathcal{U}$ and $\mathcal{T}$, the exponential laws hold, respectively $A^{B\times C} \cong (A^B)^C$ and $A^{B\wedge C} \cong (A^B)^C$ for arbitrary objects A, B and C. Let M be a topological monoid. Identify it with the one object category with morphism space M. A M-action on A is equivalent to a (continuous) monoid map $M \to \mathcal{U}(A, A)$. A basepoint preserving M-action on A is equivalent to a monoid map $M \to \mathcal{T}(A, A)$.

Given a map $A \times B \to C$, this corresponds to a map $A \to \mathcal{U}(B, C)$. Composition with a continuous functor $F : \mathcal{U} \to \mathcal{U}$ gives a map $A \to \mathcal{U}(F(B), F(C))$. Again by the exponential correspondence, this corresponds to a map $A \times F(B) \to F(C)$. Given a map $A \times B \to C$ a continuous functor $F : \mathcal{U} \to \mathcal{T}$ associates in the same way a map $A_+ \wedge F(B) \to F(C)$.

In particular a homotopy $H : I \times B \to C$ from H_0 to H_1, gives a homotopy from $F(H_0)$ to $F(H_1)$. So continuous functors respect homotopy equivalences.

There is a continuous monoid map $\vartheta : \Omega F(X) \to \widetilde{\mathcal{F}}_X(\widetilde{F}, \widetilde{F})$. The natural transformation $\vartheta(\sigma, a)$ is given at each object $f \in \mathcal{U}_X$ by

$$\vartheta(\sigma, a)(f)(e, \omega, b) \longmapsto (e, \omega * \sigma, a + b).$$

Here $(e, \omega, b) \in \widetilde{F}(f) \subset F(D(f)) \times PX$. For my notation see Section 4.1.

The $\Omega F(X)$-action on $\underline{\partial_x} F(X)$ is given by

$$\Omega F(X) \xrightarrow{\vartheta} \widetilde{\mathcal{F}}_X(\widetilde{F}, \widetilde{F}) \xrightarrow{P} \widetilde{\mathcal{E}}_X(P\widetilde{F}, P\widetilde{F}) \xrightarrow{\mathbb{K}_x} \mathcal{S}(\underline{\partial_x} F(X), \underline{\partial_x} F(X)).$$

The last two functors are also continuous as shown below, and we have a $\Omega F(X)$-action on $\underline{\partial_x} F(X)$.

Lemma 4 *$P : \mathcal{F}_X \to \mathcal{E}_X$ is a continuous functor.*

For P to be well defined, the full small subcategory $U \subset \mathcal{U}_X$ has to be closed under the suspension and the cone functors.

The functor P is continuous if for arbitrary functors $F, G \in \mathcal{F}_X$ and small $U \subset \mathcal{U}_X$ the map

$$\Pi_{f\in U}\mathcal{U}(F(f), G(f)) \cap \mathcal{F}_X(F, G) \to \Pi_{f\in U}\mathcal{U}(PF(f), PG(f)) \cap \mathcal{E}_X(PF, PG)$$

is continuous. So we need to show that

$$\Pi_{f\in U}\mathcal{U}(F(f), G(f)) \cap \mathcal{F}_X(F, G) \to \mathcal{U}(PF(g), PG(g))$$

is continuous for each $g \in U$. hocolim and holim are continuous functors (Appendix 1). Hence $\theta \mapsto P\theta(g)$ is a continuous map if each $\theta \mapsto T^n\theta(g)$ for $n \geq 0$ is a continuous map.

$$T\theta(g) = \operatorname{holim}(\theta(C_X(g)) \to \theta(S_X(g)) \leftarrow \theta(C_X(g)))$$

It follows by induction that T^n is continuous, so P is continuous. $\square$

Lemma 5 $\mathbb{K}_x : \widetilde{\mathcal{E}}_X \to \mathcal{S}$ *is a continuous functor.*

$\mathcal{S}(\underline{A}, \underline{B})$ is given the subspace topology from $\Pi_{i\geq 0}\mathcal{U}(A_i, B_i)$. Let the sets $U \subset \mathcal{U}_X$ contain s_n for all $n \geq 0$ and let them be closed under the cone and the suspension functors. $\mathbb{K}_x$ is continuous if for all functors $F, G \in \widetilde{\mathcal{E}}_X$, and for all large enough small subcategories $U \subset \mathcal{U}_X$ the following maps are continuous

$$\Pi_{f\in U}\mathcal{U}(F(f), G(f)) \cap \widetilde{\mathcal{E}}_X(F, G) \to \mathcal{U}(F_i, G_i) \quad i \geq 0.$$

Since the i-th spaces are constructed as homotopy limits of continuous maps it follows as in Lemma 4 that this is a continuous map. $\square$

2.2 ΩX-action on $\underline{\partial}_x F(X)$

We first define a geometric ΩX-action on $X \vee_x S^n$. Let R_+ be the nonnegative real numbers. Regard S^n as the one point compactification (denoted by $*$) of $R_+ \times S^{n-1}/0 \times S^{n-1}$, $n \geq 1$. The elements in S^n are denoted $[r, s]$ where $r \in R_+$ and $s \in S^{n-1}$.

We will do the constructions for general paths. Define $\Omega_{x,x'}X$ to be the space of Moore paths from x to x'. When x and x' are not in the same path component, it is empty.

The action consists of moving the spheres along the paths, and feeding out from the spheres at a rate determined by our model of the spheres. Let (ω, a) be a path from x to x'. Define the map

$$\theta(\omega, a) : X \vee_x S^n \to X \vee_{x'} S^n$$

by

$$\theta(\omega, a)|S^n[r, s] = \begin{cases} [r-a, s] & r \geq a \\ \omega(r) & a \geq r \geq 0 \end{cases}$$

and $\theta(\omega, a)|X = 1_X$. $\theta(\omega, a)$ is not a map in $\mathcal{U}_X$ from $s_{n,x}$ to $s_{n,x'}$ unless ω is a constant loop. In the cases of loop spaces we get the following monoid map

$$\Omega X \xrightarrow{\theta} \mathcal{U}(X \vee_x S^n, X \vee_x S^n).$$

The inclusions

$$S^0 \hookrightarrow S^1 \hookrightarrow \cdots S^n \hookrightarrow \cdots$$

give rise to

$$(R_+ \times S^0/0 \times S^0)^* \hookrightarrow (R_+ \times S^1/0 \times S^1)^* \hookrightarrow \cdots (R_+ \times S^n/0 \times S^n)^* \hookrightarrow \cdots$$

Similarly for sequences of inclusions of subspaces of the spheres.

Let $F : \mathcal{U} \to \mathcal{T}$ be a differentiable functor. We can give a map $\Omega_{x,x'}X \to \mathcal{T}(\widetilde{F}(s_{n,x}), \widetilde{F}(s_{n,x'}))$ by defining a homotopy $F(s_{n,x}) \simeq F(s_{n,x'}) \circ F\theta(\sigma, a)$ in a natural way. The map

$$\Omega_{x,x'}X \to \mathcal{U}(R_+, \mathcal{U}(X \vee_x S^n, X)) \times R_+$$

is defined by

$$(\sigma, a) \mapsto \begin{cases} ((t \mapsto s_{n,\sigma(a-t)} \circ \theta(\sigma_{a-t}, a-t)), a) & \text{for } t \le a \\ (s_{n,x}, a) & \text{for } t \ge a \end{cases} .$$

It is a path from $s_{n,x'} \circ \theta(\sigma, a)$ to $s_{n,x}$ of length a. σ_r is the map $t \mapsto \begin{cases} \sigma(t) & t \le r \\ \sigma(r) & t \ge r \end{cases}$. Continuity of the map follows since the adjoint map

$$\Omega_{x,x'}X \times R_+ \times (X \vee_x S^n) \to X$$

is continuous.

Composition with the functor F gives

$$F(s_{n,\sigma(a-t)}) \circ F\theta(\sigma_{a-t}, a-t) : F(X \vee_x S^n) \to F(X)$$

when $t \le a$ and $F(s_{n,x})$ when $t \ge a$. This gives a path from $F(s_{n,x'}) \circ F\theta(\sigma, a)$ to $F(s_{n,x})$.

The map $\Omega_{x,x'}X \times \widetilde{F}(s_{n,x}) \to \widetilde{F}(s_{n,x'})$ is given by

$$(\sigma, a), (y, \omega, b) \mapsto (F\theta(\sigma, a)(y), (F(s_{n,\sigma(a-t)}) \circ F\theta((\sigma_{a-t}), a-t)(y))_t * \omega, a+b).$$

The map strictly respects the composition of paths. The composite of two paths (σ, a) from x to x' and (σ', a') from x' to x'' is

$$(F\theta((\sigma * \sigma')_{a+a'-t}, a+a'-t)(y))_t = \begin{cases} (F\theta(\sigma'_{a'-t}, a'-t)(F\theta(\sigma, a)(y)))_t & 0 \le t \le a' \\ (F\theta(\sigma_{a-(t-a')}, a-(t-a'))(y))_t & a' \le t \le a+a' \end{cases}$$

which equals the composite path

$$(F\theta(\sigma'_{a'-t}, a'-t)(F\theta(\sigma, a)(y)))_t * (F\theta(\sigma_{a-t}, a-t)(y))_t.$$

In the case of loop spaces ($x = x'$) we have a monoid action: The map associated to the path $(x, 0)$ is clearly the identity, and associativity holds.

This action on $\widetilde{F}(s_n)$ extends to an action on $P\widetilde{F}(s_n)$. We use a variant of $P\widetilde{F}(s_n)$, where all elements $S_X(s_n)$, $C_X(s_n)$ are collapsed to the homotopy equivalent s_{n+1} and $(X \vee_x S^{n+1}_{\text{upper}} \to X)$ and so on. We would like to get a system of inclusions respecting the R_+ factor in the models of the spheres.

$$S^{n+k} = (R_+ \times S^{n+k-1}/0 \times S^{n+k-1})^* \quad n \ge 1, k \ge 0$$

We use standard coordinates for the $(n+k-1)$-sphere. $S^{n+k-1} = \{(x_1, \dots, x_{n+k}) \in \mathbb{R}^{n+k} \mid x_1^2 + \cdots + x_{n+k}^2 = 1\}$. Let

$$S^{n+k-1}_{i_1,\cdots,i_k} = (R_+ \times \{(x_1, \dots, x_{n+k}) \in S^{n+k-1} \mid x_{n+j} \cdot i_j \geq 0,\ 1 \leq j \leq k\}/0 \times (\bullet))^*$$

where each $i_j \in \{-1, 0, 1\}$. The first part of the diagram defining $P\widetilde{F}(s_n)$ is $\widetilde{F}$ applied to the diagram over X below.

$$\begin{tikzcd}[column sep=small]
 & & & & X \vee_x S^n \arrow[dllll] \arrow[d] \arrow[drrrr] & & & & & & \\
X \vee_x S^{n+1}_{-1} \arrow[rrrr] \arrow[dd] \arrow[drr] \arrow[dddrr] & & & & X \vee_x S^{n+1}_{0} \arrow[dd] \arrow[drr] \arrow[dddrr] & & & & X \vee_x S^{n+1}_{1} \arrow[llll] \arrow[dd] \arrow[drr] \arrow[dddrr] & & \\
 & & X \vee_x S^{n+2}_{-1,-1} \arrow[rrrr] \arrow[dll] & & & & X \vee_x S^{n+2}_{0,-1} \arrow[dll] & & & & X \vee_x S^{n+2}_{1,-1} \arrow[llll] \arrow[dll] \\
X \vee_x S^{n+2}_{-1,0} \arrow[rrrr] & & & & X \vee_x S^{n+2}_{0,0} & & & & X \vee_x S^{n+2}_{1,0} \arrow[llll] & & \\
 & & X \vee_x S^{n+2}_{-1,1} \arrow[rrrr] \arrow[ull] & & & & X \vee_x S^{n+2}_{0,1} \arrow[ull] & & & & X \vee_x S^{n+2}_{1,1} \arrow[llll] \arrow[ull]
\end{tikzcd}$$

It is a diagram of diagrams indexed by the category $(\mathbb{N}, \geq)$. To the object k is associated a 3^k diagram. The space in position $(i_1, \dots, i_k)$ in the 3^k diagram is $X \vee_x S^{n+k}_{i_1,\cdots,i_k} \to X$, where $i_j = \{-1, 0, 1\}$. The inclusions are canonical. There is a map $(i_1, \dots, i_k) \to (i'_1, \dots, i'_k)$ if and only if for each j, we have $i_j = i'_j$ or $i'_j = 0$. The action on $X \vee_x S^{n+k}_{i_1,\dots,i_k} \to X$ is defined by restricting the action on $X \vee_x S^{n+k} \to X$. We get compatible actions of $\Omega_{x,x'}X$ on the spaces in the diagram, that is an action on the diagram. Applying the continuous functor $\widetilde{F}$ gives a new diagram with $\Omega_{x,x'}X$-action. Taking homotopy limit of the 3^k diagrams gives a diagram of spaces indexed by $(\mathbb{N}, \geq)$. The diagram has an $\Omega_{x,x'}X$-action. Taking homotopy colimit (in $\mathcal{T}$) of this diagram we get a space $P'F(s_n)$ natural equivalent to $P\widetilde{F}(s_n)$ with an $\Omega_{x,x'}X$-action. $\widetilde{F}(s_n) \to P'\widetilde{F}(s_n)$ is an $\Omega_{x,x'}X$-action preserving map.

We can further extend the $\Omega_{x,x'}X$-action to an action on the derivatives. More precisely to the spectrum $\{\underline{\partial} F(X)_n\}_{n \geq 1}$, since we do not get the action defined on the zeroth space.

With the same representations of spheres, and the same remarks on suspensions and cones we have the following diagram, which is essentially the same as the diagram on page 11.

$$\begin{tikzcd}
P'\widetilde{F}(X \vee_x S^n) \arrow[r, equal] \arrow[d] & P'\widetilde{F}(X \vee_x S^n) \arrow[r] \arrow[d] & P'\widetilde{F}(X \vee_x S^n_{-1}) \arrow[d] & * \arrow[l] \arrow[d] \\
* & P'\widetilde{F}(X \vee_x S^n_1) \arrow[l] \arrow[r] & P'\widetilde{F}(X \vee_x S^{n+1}) \arrow[r, equal] & P'\widetilde{F}(X \vee_x S^{n+1}).
\end{tikzcd}$$

Since the $\Omega_{x,x'}X$-action is a basepoint preserving action, we get $\Omega_{x,x'}X$-action preserving maps

$$P'\widetilde{F}(X \vee_x S^n) \leftarrow \#_n \rightarrow \flat_n \leftarrow \Omega P'\widetilde{F}(X \vee_x S^{n+1})$$

The functor Ω preserves based actions. A based action on $A \xrightarrow{f} B$, gives an action on the diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \\ A \arrow[r, "f"] & B \\ * \arrow[r] \arrow[u] & * \arrow[u] \end{tikzcd}.$$

Taking homotopy limits in the vertical directions gives $\Omega f : \Omega A \rightarrow \Omega B$.

We here have redefined PF restricted to objects $(X \vee_x S^{n+k}_{i_1,\ldots i_k} \rightarrow X)$, and by this also the derivatives of F. What we do is to find a spectrum equivalent to the derivative, but where the ΩX-action is a strict monoid action. We could also have used the action on the derivative which is given up to homotopies, and used operades to keep track of the choices of homotopies.

A path (ω, a) from x to x' gives a map $\underline{\partial}_x F(X) \rightarrow \underline{\partial}_{x'} F(X)$, it is the identity when (ω, a) is the 0-loop. Composition with the map of a path τ from x' to x'' is the map of the composite path $\omega \circ \tau$. The map ω^{-1} gives a map $\underline{\partial}_{x'} F(X) \rightarrow \underline{\partial}_x F(X)$. The compositions $\omega \circ \omega^{-1}$ and $\omega^{-1} \circ \omega$ are (canonical) contractible loops, and the maps associated to them are equivalences. The equivalence determined by a path (ω, a) from x to x' respects (up to homotopy) the actions of the loop spaces, when the equivalence between $\Omega_x X$ and $\Omega_{x'} X$ is given by conjugation with (ω, a).

Back to the previous notation. This may justify that when $F(X)$ is path connected we left the point $y \in F(X)$ unspecified in our chain rule guess. In general we have to replace $\Omega F(X)$ with the space of paths from y to the basepoint of $F(X)$.

What does this action "see"? In a sense it picks up information about the wedge perturbations at the points in the path component of x in X, and how they are related to each other.

The action in 2.1 moves the sphere perturbation along paths in $F(X)$, in the path component of y, relating the homotopy fibers at the various points.

When the functor is the identity functor, we get two actions on the spectrum $\underline{\partial}_x \mathrm{id}(X)$. The n-th space in the spectrum and $\widetilde{\mathrm{id}}(s_n) \xrightarrow{\simeq} \Omega X_+ \wedge S^n$ are connected with natural maps which are $(2n-c)$-equivalences for n large. The actions are

$$(\sigma, a), [\omega, b; r, s] \mapsto [\omega * \sigma, a+b; r, s] \quad \text{and}$$

$$(\sigma, a), [\omega, b; r, s] \mapsto * \text{ for } r \leq a, \text{ and } [\sigma^{-1} * \omega, a+b; r-a, s] \text{ for } r \geq a.$$

So up to homotopy they are left (composed with the loop inverting automorphism) and right loop compositions. This is in agreement with our previous assumptions about the actions on $\widetilde{\mathrm{id}}(s_n)$.

Back to the chain rule guess. The guess does hold when one of the functors is the identity functor. For the case $G \circ \mathrm{id}$ we have

$$B(\underline{\partial}_y G(X)_k, \Omega X, \underline{\partial}_x \mathrm{id}(X)_\ell)$$

which is stably equivalent to $\underline{\partial}_y G(X)_k \otimes_{\Omega X} \Omega X_+ \wedge S^\ell$. This equals $\underline{\partial}_y G(X)_k \wedge S^\ell$. And in the case $\mathrm{id} \circ F$ we get

$$B(\Omega F(X)_+ \wedge S^\ell, \Omega F(X), \underline{\partial}_x F(X)_k)$$

which is equivalent to $S^\ell \wedge \underline{\partial}_x F(X)_k$. These two prespectra are equivalent to $\underline{\partial}_y G(X)$ and $\underline{\partial}_x F(X)$ respectively, upon spectrification. Note that the actions are free in these cases, so the bar constructions equals the balanced products.

2.3 Space level considerations

We will here discuss the chain rule on the space level. Let $G : \mathcal{U}_{F(X)} \to \mathcal{T}$ be a linear functor, and F a differentiable functor. First we recall some prespectra equivalent to the derivatives.

$\partial_x (G \circ F)(X)_n$ and $\mathrm{hofib}\, G \circ F(s_n)$ are $(2n - c)$-equivalent for n large. $\partial_y G(F(X))_n \sim \mathrm{hofib}(G(F(X) \vee_y S^n \to F(X)) \to G(1_{F(X)}))$. $\partial_x F(X)_n$ and $\mathrm{hofib}\, F(s_n)$ are $(2n - c')$-equivalent for n large. Here c and c' are constants.

One approach may be to describe G applied to the perturbation $F(X \vee_x S^n) \to F(X)$ by partitioning the perturbation into simpler perturbations e.g. located over contractible subspaces of X for which we have more control when applying G. Then try to to build up G on the perturbation, from G applied to these parts of the perturbation. In doing so we can expect to involve actions of $\Omega F(X)$.

For simplicity we shift notation to let $(Y \xrightarrow{f} X, r)$ be a retractive space, and let $L : \mathcal{U}_X \to \mathcal{T}$ be a linear functor.

We only need to know prespectra that are equivalent to the derivatives. The equivalences above are all respecting the loop space actions. Under spectrification the actions are preserved. This justifies that the chain rule problem can be reduced to the reconstruction of $\mathrm{hofib}\, L(f)$ from L, $\mathrm{hofib}\,(f)$, and some actions.

A linear functor on spaces over X determines for each point $x \in X$ a linear functor $\mathcal{T} \to \mathcal{T}$ as follows.

$$Z \mapsto L(X \vee_x Z \to X) \text{ and } g \mapsto L(1_X \vee_x g)$$

where the map $X \vee_x Z \to X$ is the identity on X, and the point map to x on Z. The functor is excisive since $X \vee_x Z \simeq \mathrm{hocolim}(X \leftarrow x \to Z)$ (the base points are nondegenerated). So the functor $X \vee_x : \mathcal{T} \to \mathcal{T}$ preserves cocartesian squares.

The coefficient spectrum of this linear functor $\mathcal{T} \to \mathcal{T}$ is $\mathbb{K}_x(L)$. Hence by [Good1]

$$L(X \vee_x Z \to X) \simeq \Omega^\infty(\mathbb{K}_x(L) \wedge Z)$$

at least when Z is a finite CW-complex. This generalizes to n-wedges. Define the retractive space $X \vee_{i=1}^n Z_i \xrightarrow{h} X$ by collapsing Z_i to x_i. Then

$$L(h) \simeq \prod_{i=1}^n \Omega^\infty(\mathbb{K}_{x_i}(L) \wedge Z_i).$$

This follows by induction from the following more general case. The square over X below is cocartesian (this is seen directly).

$$\begin{tikzcd} Y \vee_y Z \arrow[r, "f \vee 1_Z"] \arrow[d] & X \vee_{f(y)} Z \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Hence

$$L(Y \vee_y Z \to X) \xrightarrow{\simeq} \mathrm{holim}(L(f) \to L(1_X) \leftarrow L(X \vee_{f(y)} Z \to X) \xrightarrow{\simeq}$$
$$L(f) \times L(X \vee_{f(y)} Z \to X).$$

In this case it seems that we get a product (over X) of fibers. One may think that in the general case, one should have an "integral" over X. It is not clear what this should be. One idea is a Čech like construction of something like a sheaf with values in infinite loop spaces (spectra).

A candidate is the covariant functor from a properly defined category of subspaces of X, defined by $U \mapsto L(f^{-1}(U) \cup r(X) \to X)$. The category could e.g. have as objects closed contractible subspaces of X whose inclusion maps to X are cofibrations, and inclusions as morphisms. Another possibility is to let X be a simplicial complex and let U run over its simplices. I will not try to develop this idea further.

We can replace $(Y \xrightarrow{f} X, r)$ by an equivalent retractive space over X, the mapping path space (cf.p 33). This is a fibration, and hence a path in X gives an equivalence between the fibers at the start and the end points of the path. The ΩX-actions are compatible when the equivalence between the loop spaces is given by conjugation with the path determining the equivalence. So here it would be enough to know fibers at each path component. The fibers of the mapping path fibration of f are just the homotopy fibers of f.

In the n-wedge case above, the homotopy fiber at x is

$$P_x X \cup_i \Omega_{x_i,x} X \times Z_i \xrightarrow{\simeq} \vee_i (\Omega_{x_i,x} X)_+ \wedge Z_i$$

Back to the chain rule setting. Assume the perturbation $F(s_n)$ of $1_{F(X)}$ is a finite wedge perturbation with Z_i attached at $y_i \in F(X)$.

$$F(X \vee_x S^n) \simeq F(X) \vee_{i=1}^n Z_i \to F(X)$$

Then

$$\underline{\partial}_x F(X)_n \overset{2n}{\sim} \vee_{i=1}^n \Omega_{y,y_i} F(X)_+ \wedge Z_i \text{ and}$$

$$\Omega^\infty \underline{\partial}_y G(F(X))_n \overset{2n}{\sim} \prod_{i=1}^n \Omega^\infty(\underline{\partial}_{y_i} G(F(X)) \wedge Z_i).$$

Here $\overset{2n}{\sim}$ means a chain of maps which are approximately $2n$-equivalences. When $F(X)$ is path connected the last space is equivalent to

$$\Omega^\infty(\underline{\partial}_y G(F(X)) \wedge \vee_{i=1}^n Z_i) \overset{2n}{\sim} \Omega^\infty(\underline{\partial}_y G(F(X)) \wedge_{\Omega F(X)} \underline{\partial}_x F(X)_n).$$

Here we use that the derivatives $\underline{\partial}_{y_i} G(F(X))$, $i = 1, \dots n$ are all equivalent since the points y_i are in the same path component. So the chain rule guess holds in this special case. In this case we do not get the actions of the derivatives of G involved. This can be explained by the simplicity of the perturbation.

The first approach I tried was to build a spectral sequence with input hofib(γ), ΩX and ΩX-action on the homotopy fiber, converging to hofib $L(\gamma)$. I have only been able to set up this in a satisfying way when L is a linear functor $L : \mathcal{U} \to \mathcal{T}$. In the chain rule setting this corresponds to $F(X)$ contractible. This case has already been explained in Proposition 3, and we also have a explicit expression for it in a result of Stasheff. A detailed treatment of this spectral sequence, and my attemps to extend it to the setting of spaces over X, is given in Sections 4.3 and 4.5. I will discuss the relevant part of it here.

The spectral sequence could at least show what information that is needed in order to reconstruct hofib $L(\gamma)$. It could also have happened that the spectral sequence in some useful cases collapsed at the E^2-term. The action first appears in the d^1-differential, so we should not expect it to collapse at the E^1-term. The primary hope was that the spectral sequence could be compared with other spectral sequences, associated to a guess of the chain rule. We then would have had to get maps (or sequences of maps) between hofib $L(\gamma)$ and the guess, respecting the exact couples associated to them. This would induce a map of the spectral sequences. If this map had induced isomorphisms between all the $E^1_{p,q}$-modules and respected all the differentials, the guess would have been hofib $L(\gamma)$. In particular if the spectral sequence collapsed at the E^2-term, we would only have to compare the d^1-differentials.

The guess I thought of was a bar construction of hofib $L(\gamma)$ and the coefficient spectra of L over the loop space of X. At least in the case when X is contractible. In the more general cases it is not so clear how the derivatives of L would appear.

In these approaches it would have been essential to know the d^1-differential. I spent some effort describing the d^1-differential of the spectral sequence in the case when X is a point. I was not able to describe the E^2-term explicitly or as a known construction like Tor.

An easy diagram argument shows that there is a sequence of natural equivalences between hofib $L(\gamma)$ and $\Omega L(\text{hocofib}(\gamma))$ (cf. Lemma 31). The homotopy cofiber is taken in the category of spaces over X. From this it seems that we do not need to get actions of the loop spaces of X involved. In the spectral sequence approach we are given the whole functor L. In the chain rule problem we are only given the derivatives of L at various points of X. And here we expect to get involved actions on the coefficient spectra of L. This may be explained by that reconstructing the derivative of $L : \mathcal{U} \to \mathcal{T}$ from all the derivatives at X would involve more information: loop space actions of X.

State of affairs. The discussion ends here. The goal is not reached, namely to give an expression, or at least a spectral sequence telling what the derivative of a composite functor $G \circ F$ is in terms of the derivatives of F and G. I hope the discussion here has been a step in the right direction. It could also have been helpful to know more examples, for which one could test the chain rule guesses.

3 Point set topology

Let $\mathcal{H}$ be the category of all Hausdorff spaces, and all continuous maps. Let $\mathcal{U}$ be the category of all compactly generated spaces, and all continuous maps, and let $\mathcal{T}$ be the category of all compactly generated based spaces, and basepoint preserving maps. $\mathcal{W}$ is the category of spaces of a homotopy type of a CW-complex. Let $\mathcal{U}_X$ be the overcategory, consisting of compactly generated spaces over a compactly generated space X (cf.p. 7). A space is called compactly generated if it is Hausdorff, and if any subset which is open in each compact subset is itself open. There is a compactly generating functor $\mathrm{k} : \mathcal{H} \to \mathcal{U}$. Given a space $X \in \mathcal{H}$ then k(X) has the same point set as X, the open sets of $\mathrm{k}(X)$ are all the subsets of X which are open in each compact subsets of X.

Properties of k, let $X, Y \in \mathcal{H}$:

1. $\mathrm{id} : \mathrm{k}(X) \to X$ is a continuous map.
2. $\mathrm{k}(X)$ is a Hausdorff space.
3. $\mathrm{k}(X)$ and X have the same compact sets.
4. $\mathrm{k}(X)$ is compactly generated.
5. If X is compactly generated, then $\mathrm{k}(X) = X$.
6. If $f : X \to Y$ is continuous on compact spaces, then $\mathrm{k}(f) : \mathrm{k}(X) \to \mathrm{k}(Y)$ is a continuous map.
7. $\mathrm{k}(X \times Y) = \mathrm{k}(\mathrm{k}(X) \times \mathrm{k}(Y))$.
8. k is a right adjoint of the functor $\mathcal{U} \hookrightarrow \mathcal{H}$.

The induced topology on X from $X \overset{p_\alpha}{\to} X_\alpha$ is the smalest topology on X so that each p_α is a continuous map. The coinduced topology on X from $X_\alpha \overset{q_\alpha}{\to} X$ is the largest topology on X so that each q_α is a continuous map.

We will work in $\mathcal{U}$, when constructing spaces from other we will apply k afther each step, we have to be careful not to leave $\mathcal{H}$.

In lemma 6 to 9 we assume that all the spaces are Hausdorff.

Lemma 6 *X get the same topology from $X \overset{p_\alpha}{\to} X_\alpha$ and $X \overset{p_\alpha}{\to} \mathrm{k}(X_\alpha)$.*

Lemma 7 *Let X get the induced topology from $X \overset{p_\alpha}{\to} X_\alpha$ then $f : Y \to X$ is a continuous map $\Leftrightarrow$ each $p_\alpha \circ f$ is a continuous map.*

Lemma 8 *Compositions of induced topologies is an induced topology.*

Lemma 9 *Let X get the coinduced topology from $X_\alpha \overset{q_\alpha}{\to} X$, if each $X_\alpha \in \mathcal{U}$ and $Y \in \mathcal{U}$, then a map $X \overset{f}{\to} Y$ is continuous $\Leftrightarrow$ each $f \circ q_\alpha$ is continuous.*

Lemma 10 *In $\mathcal{U}$ (start with spaces in $\mathcal{U}$) compositions of coinduced topologies is a coinduced topology.*

Lemma 11 *If $X_i \xrightarrow{q_i} Y_i$ gives coinduced topology on Y_i and the map q_i is surjective (q_i is then called an identification), here $i = 1, 2$, then the product topology on $Y_1 \times Y_2$ is the same as the coinduced topology from $q_1 \times q_2$.*

Lemma 12 *Let X get the induced topology from $X \xrightarrow{p_\alpha} X_\alpha$. If each $X_\alpha \in \mathcal{H}$ and $\{p_\alpha\}$ separates points, then $X \in \mathcal{H}$.*

Corollary 13 *In $\mathcal{U}$ cartesian product is a categorical product, and disjoint union is a categorical sum.*

Lemma 14 *For $A, B \subset X$ both open or closed, then $A \cup B$ gets the same topology from $A \cup B \hookrightarrow X$ and $A \amalg B \to A \cup B$.*

Lemma 15 *Let $p : Y \to X$ be an identification and let $A \subset X$ be an open or a closed subspace of X. Then the induced topology on A from $A \hookrightarrow X$ is the same as the coinduced topology on A from $p^{-1}(A) \xrightarrow{p} A$ where $p^{-1}(A)$ gets the induced topology from $p^{-1}(A) \hookrightarrow Y$.*

Lemma 16 *Let $A \subset X$ be a closed subset, then CA gets the same topology from $CA \hookrightarrow CX$ and $A \times I \to CA$.*

Lemma 17 *If Y is a Hausdorff space then so is CY, and for arbitrary X, $\mathcal{U}(X, Y)$ is a Hausdorff space.*

Lemma 18 *If X and Y are Hausdorff spaces, then $X * Y$ is a Hausdorff space.*

One of the main reasons why $\mathcal{U}$ is a convenient category of topological spaces (cf. [Steen]) is that it behaves well on morphisms spaces. It is suggestive to write $\mathcal{U}(X, Y)$ as Y^X. We have the following lemmas in the category of compactly generated spaces.

Lemma 19 *The evaluation map $Y^X \times X \to Y$ is a continuous map.*

Lemma 20 *Compositions of maps $Z^Y \times Y^X \to Z^X$ is a continuous map.*

Lemma 21 *The exponential correspondences*

$$Z^{Y \times X} \cong (Z^Y)^X \text{ and } (Y \times Z)^X \cong Y^X \times Z^X.$$

4 A spectral sequence

In this section it may be helpful if the reader draws pictures of the paths, in order to visualize what happens.

4.1 Moore loops

Let R_+ be all the nonnegative real numbers, and let $B \in \mathcal{U}$. The space of Moore paths is $N = \{(\omega, a) \in \mathcal{U}(R_+, B) \times R_+ \mid \omega(x) = \omega(a) \text{ for } x \geq a\}$.

The space of ordered pairs of Moore paths, such that the first path ends where the second path starts is $M = \{(\omega, a, \omega', a') \in \mathcal{U}(R_+, B) \times R_+ \times \mathcal{U}(R_+, B) \times R_+) \mid \omega(x) = \omega(a) \text{ for } x \geq a\ ,\ \omega'(x) = \omega'(a') \text{ for } x \geq a' \text{ and } \omega(a) = \omega'(0)\}$.

Maps. Multiplication $m : M \to N$, inverse $i : N \to N$, and two maps which cut off paths $n_1 : N \times R_+ \to N$ and $n_2 : N \times R_+ \to N$ are given by: $m(\omega, a, \omega', a') = (\omega * \omega', a + a')$ where

$$\omega * \omega'(t) = \begin{cases} \omega(t) & 0 \leq t \leq a \\ \omega'(t-a) & a \leq t. \end{cases}$$

$i(\omega, a) = (\omega^{-1}, a)$ where

$$\omega^{-1}(t) = \begin{cases} \omega(a-t) & 0 \leq t \leq a \\ \omega(0) & a \leq t. \end{cases}$$

$n_1(\omega, a, r) = (\omega_r, r)$ where

$$\omega_r(t) = \begin{cases} \omega(t) & 0 \leq t \leq r \\ \omega(r) & r \leq t. \end{cases}$$

$n_2(\omega, a, r) = (\omega^r, r)$ where

$$\omega^r(t) = \begin{cases} \omega(\max(0, a-r)) & 0 \leq t \leq \max(0, r-a) \\ \omega(t + a - r) & \max(0, r-a) \leq t \leq r \\ \omega(a) & r \leq t. \end{cases}$$

Lemma 22 *m is a continuous map.*

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "m"] & \mathcal{U}(R_+, B) \times R_+ \times \mathcal{U}(R_+, B) \times R_+ \\ N \arrow[r, hook] & \mathcal{U}(R_+, B) \times R_+ \end{tikzcd}$$

We will show that composition to $\mathcal{U}(R_+, B) \times R_+$ is continuous (before compact generating). Composition with projection to R_+ is continuous. Choose an element (ω, a, ω', a') in M, and an open (generator) neighborhood

$[K,U]$ of $\omega * \omega'$, where K is a compact set in R_+, and U is a open set in B. Then

$$(\omega)^{-1}(U) \text{ open} \supset K \cap [0,a] \text{ and}$$
$$(\omega')^{-1}(U) \text{ open } \supset [0,\infty\rangle \cap (K-a).$$

Here $(K-a)$ is the subspace $\{\lambda \in \mathbb{R} \mid \lambda + a \in K\}$. There exist a $\varepsilon_1 > 0$ so that $(\omega)^{-1}(U) \supset K \cap [0, a+\varepsilon_1]$. For each point k in $[0,\infty\rangle \cap (K-a)$, there is a $3\varepsilon_k$-neighborhood contained in $(\omega')^{-1}(U)$. Since $[0,\infty\rangle \cap (K-a)$ is compact it is covered by finitely many of the ε_k-neighborhoods, say at $k_1,\ldots,k_n$. Let $\varepsilon_2 = \min(\varepsilon_{k_1},\ldots,\varepsilon_{k_n})$, then

$$(\omega')^{-1}(U) \supset (\cup_{|\lambda|\leq\varepsilon_2}(K-a+\lambda)) \cap [0,\infty\rangle.$$

This is again a compact set. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

$$(\omega, a, \omega', a') \in [K \cap [0, a+\varepsilon]\,, U] \times (a-\varepsilon, a+\varepsilon)\times$$
$$[\,(\cup_{|\lambda|\leq\varepsilon_2}(K-a+\lambda)) \cap [0,\infty\rangle, U] \times (a'-\varepsilon, a'+\varepsilon) \cap M.$$

This maps into $[K,U]$, so projection to $\mathcal{U}(R_+, B)$ is also continuous, hence m is a continuous map. $\square$

Lemma 23 *i is a continuous map.*

$$\begin{tikzcd}
N \arrow[r, hook] \arrow[d, "i"'] & \mathcal{U}(R_+,B) \times R_+ \arrow[d, "i'"] \\
N \arrow[r, hook] & \mathcal{U}(R_+,B) \times R_+
\end{tikzcd}$$

i is continuous if the composition of i' with the inclusion is continuous. Projection to R_+ is a continuous map. Choose an element (ω, a) in N, and an open (generator) neighborhood $[K,U]$ of ω^{-1}. If $K \cap [a,\infty\rangle = \emptyset$ then $\omega \in [(a-K), U]$, since K is closed there is an $\varepsilon_1 > 0$ so $K \cap [a-\varepsilon_1, \infty\rangle = \emptyset$, $(\omega)^{-1}(U)$ open $\supset a - K$, so there exist an $\varepsilon_2 > 0$ so

$$(\omega)^{-1}(U) \supset (\cup_{|\lambda|\leq\varepsilon_2}(K-a+\lambda)) \cap [0,\infty\rangle.$$

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

$$(\omega, a) \in [\,(\cup_{|\lambda|\leq\varepsilon_2}(K-a+\lambda)) \cap [0,\infty\rangle, U] \times (a-\varepsilon, a+\varepsilon) \cap [0,\infty\rangle$$

maps into $[K,U]$. If $K \cap [\,a,\infty\rangle \neq \emptyset$ then $\omega(0) \in U$, $(\omega)^{-1}(U)$ open $\supset$ $(a-K) \cap [0,\infty\rangle$, hence there exist an $\varepsilon > 0$ so

$$(\omega)^{-1}(U) \supset (\cup_{|\lambda|\leq\varepsilon_2}(K-a+\lambda)) \cap [0,\infty\rangle$$

$$(\omega, a) \in [\,\{0\} \cup (\cup_{|\lambda|\leq\varepsilon_2}(K-a+\lambda)) \cap [0,\infty\rangle, U] \times (a-\varepsilon, a+\varepsilon) \cap [0,\infty\rangle$$

and this maps into $[K,U]$, hence i is a continuous map. $\square$

Lemma 24 *The maps n_1 and n_2 are continuous.*

$$\begin{tikzcd} N \times R_+ \arrow[r, hook] \arrow[d, "n_1"] & \mathcal{U}(R_+, B) \times R_+ \times R_+ \arrow[d] \\ N \arrow[r, hook] & \mathcal{U}(R_+, B) \times R_+ \end{tikzcd}$$

Composition with the projection to R_+ is continuous. We will show that the composition of the map $n_1 : (\omega, a, r) \longmapsto \omega_r, r$ with the projection to $\mathcal{U}(R_+, B)$ is continuous.

Choose an element (ω, r) in $\mathcal{U}(R_+, B) \times R_+$, and an open (generator) neighborhood $[K, U]$ of ω_r.
If $[r, \infty\rangle \cap K = \emptyset$ then there is an $\varepsilon > 0$ so $(r - \varepsilon, r + \varepsilon) \cap K = \emptyset$ so

$$(\omega, r) \in [K, U] \times (r - \varepsilon, r + \varepsilon) \cap [0, \infty\rangle$$

which maps into $[K, U]$. If $[r, \infty\rangle \cap K \neq \emptyset$ then $\omega(r) \in U$. $r \in \text{open}(\omega)^{-1}(U)$ hence there exists an $\varepsilon > 0$ so that $[r - \varepsilon, r + \varepsilon] \subset (\omega)^{-1}(U)$, so

$$(\omega, r) \in [K \cup [r - \varepsilon, r + \varepsilon], U] \times (r - \varepsilon, r + \varepsilon)$$

and this maps into $[K, U]$, hence n_1 is continuous.
$n_2 = i \circ n_1 \circ i \times 1_{R_+}$, hence n_2 is continuous. $\square$

Moore path space, given $b_0 \in B$.

$$P_{b_0}B = \{(\sigma, a) \in \mathcal{U}(R_+, B) \times R_+ \mid \sigma(a) = b_0 \text{ and } \sigma(x) = \sigma(a) \text{ for } x \geq a \}.$$

The projection to B, given by $\sigma, a \mapsto \sigma(0)$ is continuous. This is a fibration and the fiber at b_0 is the Moore loop space at b_0.

$$\Omega_{b_0}B = \{(\sigma, a) \in \mathcal{U}(R_+, B) \times R_+ \mid \sigma(0) = \sigma(a) = b_0 \text{ and } \sigma(x) = \sigma(a) \text{ for } x \geq a\}.$$

Let $\mu = m \mid \Omega_{b_0}B \times \Omega_{b_0}B$ and $\iota = i \mid \Omega_{b_0}B$.

$$\begin{tikzcd} \Omega_{b_0}B \times \Omega_{b_0}B \arrow[r, hook] \arrow[d, "\mu"] & M \arrow[d, "m"] \\ \Omega_{b_0}B \arrow[r, hook] & N \end{tikzcd} \qquad \begin{tikzcd} \Omega_{b_0}B \arrow[r, hook] \arrow[d, "\iota"] & N \arrow[d, "i"] \\ \Omega_{b_0}B \arrow[r, hook] & N \end{tikzcd}$$

So μ and ι are continuous. The element $(b_0, 0) \in \Omega_{b_0}B$ where b_0 is the point map, is an identity element for μ. The map μ is associative and ι is a homotopy inverse.

$$\begin{tikzcd} \Omega_{b_0}B \times I \arrow[rrrr] \arrow[d, hook] & & & & \Omega_{b_0}B \arrow[d, hook] \\ N \times I \arrow[r, "\alpha"] & N \times R_+ \arrow[r, "n_1"] & N \arrow[r, "1_N \times i \circ \text{diag}"] & M \arrow[r, "m"] & N \end{tikzcd}$$

The map α is defined by $\alpha(\omega, a, t) = (\omega, a, a \cdot t)$. This gives a homotopy from $(b_0, 0)$ to $\mu \circ (1 \times \iota) \circ \text{diag}$. Similarly $\mu \circ (\iota \times 1) \circ \text{diag}$ is homotopic to $(b_0, 0)$.

Lemma 25 *$\Omega_{b_0}B$ is a grouplike topological monoid.* $\square$

Mapping path space. Given an arbitrary map (the notation will be convenient later)

$$\overline{C} \xrightarrow{\overline{r}} B$$

let

$$C = \{(\overline{c}, \omega, a) \in \overline{C} \times \mathcal{U}(R_+, B) \times R_+ \mid \overline{r}(\overline{c}) = \omega(0), \omega(x) = \omega(a) \text{ for } x \geq a\}.$$

Define the following maps

$$\begin{aligned} r &: C \to B \text{ by } (\overline{c}, \omega, a) \mapsto \omega(a) \\ s &: \overline{C} \to C \text{ by } \overline{c} \mapsto (\overline{c}, \overline{r}(\overline{c}), 0) \\ t &: C \to \overline{C} \text{ by } (\overline{c}, \omega, a) \mapsto \overline{c}. \end{aligned}$$

Lemma 26 *The maps r, s and t are continuous, and $\overline{C}$ is a strong deformation retract of C.*

For a point $b \in B$, we will also denote the point map in $\mathcal{U}(R_+, B)$ which maps R_+ to b, by b.

$$r: C \hookrightarrow \overline{C} \times \mathcal{U}(R_+, B) \times R_+ \xrightarrow{\text{proj}} \mathcal{U}(R_+, B) \times R_+ \xrightarrow{\text{ev}} B.$$

r is continuous since the evaluation map $\text{ev}(\omega, a) = \omega(a)$ is continuous.

$$\overline{C} \xrightarrow{s} C \hookrightarrow \overline{C} \times \mathcal{U}(R_+, B) \times R_+$$

s is a continuous map if the composition with each projection is continuous. The projections to $\overline{C}$ and R_+ are continuous.

$$\begin{tikzcd} \overline{C} \arrow[rr] \arrow[dr, "\overline{r}"'] & & \mathcal{U}(R_+, B) \\ & B \arrow[ur] & \end{tikzcd}$$

The map $B \to \mathcal{U}(R_+, B)$, given by $b \mapsto b$ is continuous.

$t \circ s = 1_{\overline{C}}$ and $s \circ t\ (\overline{c}, \omega, a) = (\overline{c}, \overline{r}(\overline{c}), 0)$ is homotopic to 1_C.

$$\begin{tikzcd} C \times I \arrow[r, hook] \arrow[d] & \overline{C} \times N \times \mathcal{U}(R_+, R_+) \arrow[d, "1_{\overline{C}} \times n_1"] & \\ C \arrow[r, hook] & \overline{C} \times N \arrow[r, hook] & \overline{C} \times \mathcal{U}(R_+, B) \times R_+ \end{tikzcd}$$

This gives a homotopy $1_{\overline{C}} \times n_1 | C \times I : s \circ t \simeq 1_C \operatorname{rel} \overline{C}$. So $\overline{C}$ is a strong deformation retract of C. $\square$

Lemma 27 *$r : C \longrightarrow B$ is a fibration.*

Given

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, hook] & C \arrow[d, "r"] \\ W \times I \arrow[r, "G"] \arrow[ur, dashed, "G'"] & B \end{tikzcd}$$

The composite map $W \xrightarrow{g} C \hookrightarrow \overline{C} \times \mathcal{U}(R_+, B) \times R_+$ is (g_1, g_2, g_3), where each g_i is a continuous map. By commutativity of the square $G(w,0) = g_2(w)(g_3(w))$. Since (g_1, g_2, g_3) maps into C we have $\overline{r}(g_1(w)) = g_2(w)(0)$. Define $G' : W \times I \to C$ by (g^1, g^2, g^3). G' is a continuous map if g^1, g^2 and g^3 are continuous maps. Define $g^1 = g_1$, $g^2(w,t) = g_2 * G(w)_t$ and $g^3(w,t) = g_3(w,t) + t$. Here we regard G as the map $G : W \to B^I$. We have $g^i(w,0) = g_i(w,0)$ for $i = 1,2,3$. $G(w,t) = g^2(w,t)(g^3(w,t))$ and $\overline{r}g^1(w,t) = g^2(w,t)(0)$. So G' is a lifting of G if it is continuous. g^1 and g^3 are continuous.

$$W \times I \to M \hookrightarrow \mathcal{U}(R_+, B) \times R_+ \times \mathcal{U}(R_+, B) \times R_+$$

$$(w,t) \mapsto (g_2(w), g_3(w), G(w, \cdot)_t, t).$$

This is a continuous map. So $W \times I \to M \xrightarrow{m} N \to \mathcal{U}(R_+, B)$ which is g^2, is a continuous map. $\square$

Special cases of homotopy limits. The homotopy colimit of the diagram

$$(X_2 \xleftarrow{f_2} X_{12} \xrightarrow{f_1} X_1)$$

is $X_2 \amalg X_{12} \times I \amalg X_1 / f_2(x_{12}) \sim x_{12} \times 0$ and $x_{12} \times 1 \sim f_1(x_{12})$
$\mathrm{hocofib}(X_{12} \xrightarrow{f_1} X_1)$ is the homotopy colimit of $(* \longleftarrow X_{12} \xrightarrow{f_1} X_1)$.

The homotopy limit of the diagram

$$(X_2 \xrightarrow{g_2} X \xleftarrow{g_1} X_1)$$

is $\{\ (x_2, \omega, a, x_1) \in X_2 \times N \times X_1 \mid \omega(0) = g_1(x_1) \text{ and } \omega(a) = g_2(x_2)\ \}$.
$\mathrm{hofib}_{x_0}(X_1 \xrightarrow{g_1} X)$, given a basepoint $x_0 \in X$, is the homotopy limit of $(x_0 \hookrightarrow X \xleftarrow{g_1} X_1)$. The limits are easily seen to be Hausdorff spaces when X, X_1, X_2, and X_{12} are Hausdorff spaces.

The definition of homotopy limit given here (in this special case) differs from that given in Appendix 1, the usual, which is defined by using paths of indexing length 1 instead of Moore paths. The one here is more convenient, since we get a monoid action on it.

Lemma 28 *The two definitions of homotopy limit for the diagram $(X_2 \xrightarrow{g_2} X \xleftarrow{g_1} X_1)$ (in $\mathcal{T}$) are homotopy equivalent. In the case of loop spaces the equivalence respects the H-group structures, and they strictly respect the identities. In the case of homotopy fibers the equivalence respects the canonical loop space actions, up to homotopy.*

Let x_0 be the basepoint of X. Since the point map $x_0 \in \mathcal{U}(I,X)$ is nondegenerated there exists a map $\eta : \mathcal{U}(I,X) \to R_+$ (the nonnegative real numbers) so that $\eta^{-1}(0) = x_0$. Define the map from the usual homotopy limit by

$$(x_2, \omega, x_1) \mapsto \begin{cases} (x_2, (t \mapsto \omega(t/\eta(\omega)), \eta(\omega)\,), x_1) & \omega \neq x_0 \\ (x_2, (t \mapsto x_0, 0\,), x_1) & \omega = x_0 \end{cases}.$$

Define the map from the Moore homotopy limit by

$$(x_2, \sigma, a, x_1) \mapsto (x_2, t \mapsto \sigma(a \cdot t), x_1).$$

Composition in this order is the identity, in the reversed order it is

$$(x_2, \sigma, a, x_1) \mapsto \begin{cases} (x_2, (t \mapsto \sigma(a \cdot t/\eta(\sigma(a \cdot))), \eta(\sigma(a \cdot))\,), x_1) & \eta(\sigma(a \cdot)) \neq 0 \\ (x_2, (t \mapsto x_0, 0\,), x_1) & \eta(\sigma(a \cdot)) = 0 \end{cases}$$

this is homotopic to the identity. $\square$

4.2 A homotopy fiber

$$\begin{tikzcd}
\mathrm{hofib}_{b_0}(\overline{r}_1) \arrow[r] & \overline{C}_1 \arrow[r, "\overline{r}_1"] & B \\
F \arrow[r, hook, "i"] \arrow[dr, bend right, "t"'] & C \arrow[u, "j"] \arrow[r, "r"] & B \arrow[u, equal] \\
 & \overline{C} \arrow[u, "s"] \arrow[r, "\overline{r}"] & B \arrow[u, equal]
\end{tikzcd}$$

The lower square is as above. Definitions: $F = r^{-1}(b_0)$, $\overline{C}_1 = \mathrm{hocofib}(i) = CF \cup C$, it is a Hausdorff space. $\overline{r}_1|CF = b_0$ is a point map, and $\overline{r}_1|C = r$. The map $\mathrm{hofib}_{b_0}(\overline{r}_1) \to \overline{C}_1$ is given by $(\overline{c}, \omega, a) \mapsto \overline{c}$. (Note the notation is a bit sloppy, C denotes both the cone functor and the space C.)

$\overline{r}_1$ is continuous by the diagram

$$\begin{tikzcd}
 & CF \cup C \arrow[r, "\overline{r}_1"] & B \\
CF \arrow[r, hook] & CF \amalg C \arrow[u, two heads] & C. \arrow[l, hook']
\end{tikzcd}$$

The following proposition is based upon a result of Ganea. With our added hypothesis that the fibration in question is a mapping path fibration we get strong control of the homotopy equivalence w. We are able to give an explicit map rather than merely show that the map exists. This will be useful in 4.4. In fact our consideration is not less general then that of Ganea, since every (quasi-)fibration factorizes through a mapping path fibration, with weak equivalences between the fibers. An easy argument using the weak invariance of homotopy (co)fibers shows that when $\overline{r}$ is a quasi-fibration there is a sequence of weak equivalences connecting $\mathrm{hofib}_{b_0}(\mathrm{hocofib}(\overline{r}^{-1}(b_0) \to \overline{C}) \to B)$ which appears in Ganea's result, and $\overline{r}^{-1}(b_0) * \Omega B$.

Proposition 4 *If $B \in \mathcal{W}$, then there is a homotopy equivalence* $w : F * \Omega_{b_0} B \to \mathrm{hofib}_{b_0}(\overline{r}_1)$.

We will prove this by factoring the Ganea map w into three maps, and show that each map is a homotopy equivalence. I will omit reference to the basepoints needed for path and loop spaces and for homotopy fibers in the remainder of this section. The join is defined as $X * Y = \mathrm{hocolim}(X \leftarrow X \times Y \to Y)$, elements are on the form $(1-s)x + sy$, the equivalence class of (x, s, y), where $0x = 0x'$ for all x and x', and similarly $0y = 0y'$. Similar notations are used for cones, an element in CX is written sx, and for suspensions, an element in SX is written $sx/1(X)$.

$$F * \Omega B \xrightarrow{w_1} CF \times \Omega B \cup F \times C\Omega B \xrightarrow{w_2} CF \times \Omega B \cup F \times PB$$
$$\xrightarrow{w_3} \mathrm{hofib}(\overline{r}_1) = CF \times \Omega B \cup \mathrm{hofib}(r)$$

w_1 is independent of the given spaces, so I give it for X and Y. $w_1 : X * Y \to CX \times Y \cup X \times CY$ is given by

$$w_1((1-s)x + sy) = \min(1, 2-2s)x + \min(1, 2s)y.$$

$$\begin{tikzcd}
X \times Y \times I \arrow[r] \arrow[d] & X \times Y \times I \cup X \times Y \times I \arrow[r] \arrow[d] & X \times Y \times I \arrow[d] \\
X * Y \arrow[r, "w_1"] & CX \times Y \cup X \times CY \arrow[r, "(w_1)^{-1}"] & X * Y
\end{tikzcd}$$

The vertical maps are identifications, hence w_1 is a homeomorphism.

$\Phi : (C\Omega B, \Omega B) \to (PB, \Omega B)$, given by $s(\omega, a) \longmapsto (\omega^{as}, as)$, is continuous by the diagram

$$\begin{tikzcd}
\Omega B \times I \arrow[r, two heads] \arrow[dr, hook] & C\Omega B \arrow[r, "\Phi"] & PB \arrow[r, hook] & N. \\
& N \times R_+ \arrow[urr, "n_2"'] & &
\end{tikzcd}$$

Define w_2 by $w_2(s(\overline{c}, \omega, a) + t(\omega', a')) = s(\overline{c}, \omega, a) + \Phi(t(\omega', a'))$. w_2 is continuous by the diagram

$$\begin{tikzcd}
CF \times \Omega B \cup F \times C\Omega B \arrow[r, "w_2"] \arrow[d, hook] & CF \times \Omega B \cup F \times PB \arrow[d, hook] \\
CF \times C\Omega B \arrow[r, "1_{CF} \times \Phi"] & CF \times PB.
\end{tikzcd}$$

Φ is a relative homotopy equivalence. $C\Omega B$ and PB are contractible. $(PB, \Omega B)$ is homotopy equivalent to the usual path and loop space pair (shown on page 34). By [Milnor], the usual ΩB is of the homotopy type of a CW-complex, and the usual $(PB, \Omega B)$ is of the homotopy type of a relative CW-complex. So both range and domain of Φ are of the homotopy type of a relative CW-complex. This makes it possible to extend maps and homotopies cell by cell, to show that Φ is a relative homotopy equivalence. The details are given in Appendix 2. Let Ψ be a relative homotopy inverse of Φ.

Define $\tilde{w}_2 : CF \times \Omega B \cup F \times PB \longrightarrow CF \times \Omega B \cup F \times C\Omega B$ by $\tilde{w}_2(s(\overline{c}, \omega, a) \times (\omega', a')) = s(\overline{c}, \omega, a) \times \Psi(\omega', a')$.
By the same diagram as for w_2 with the horizontal arrows reversed, it follows that $\tilde{w}_2$ is continuous.

If $H : \Psi \circ \Phi \simeq 1_{C\Omega B}$ then

$$\begin{tikzcd}
CF \times \Omega B \cup F \times C\Omega B \times I \arrow[r] \arrow[d, hook] & CF \times \Omega B \cup F \times C\Omega B \arrow[d, hook] \\
CF \times C\Omega B \times I \arrow[r, "1_{CF} \times H"] & CF \times C\Omega B
\end{tikzcd}$$

so $\tilde{w}_2 \circ w_2 \simeq 1$, similarly $w_2 \circ \tilde{w}_2 \simeq 1$ and w_2 is a homotopy equivalence.

$$w_3 \; : \; CF \times \Omega B \cup F \times PB \longrightarrow CF \times \Omega B \cup \mathrm{hofib}(r)$$

is given by $w_3(s(\overline{c}, \omega, a), \sigma, b) = (s(\overline{c}, \omega * \sigma^{-1}, a + b), \sigma, b)$. (Note $\mathrm{hofib}(r)$ is closed in $\overline{C} \times M$ since B is Hausdorff.)

$$\begin{tikzcd}
CF \times \Omega B \cup F \times PB \arrow[r, "w_3"] & CF \times \Omega B \cup \mathrm{hofib}(r) \\
I \times F \times \Omega B \cup F \times PB \arrow[r] \arrow[u, two heads] \arrow[d, hook] & I \times F \times \Omega B \cup \mathrm{hofib}(r) \arrow[u, two heads] \arrow[d, hook] \\
I \times \overline{C} \times N \times N \arrow[r] & I \times \overline{C} \times M
\end{tikzcd}$$

w_3 is continuous if the composition in the lower square is continuous, it is

$$1 \times m \times 1_N \circ 1 \times i \times 1_N \circ 1 \times \mathrm{diag} \times 1_N \circ \mathrm{inclusion}$$

so w_3 is continuous. We define a homotopy inverse of w_3

$$v \; : \; CF \times \Omega B \cup \mathrm{hofib}(r) \longrightarrow CF \times \Omega B \cup F \times PB.$$

by $v(s(\overline{c}, \omega, a), \sigma, b) = (s(\overline{c}, \omega * \sigma, a + b), \sigma, b)$. v is continuous by a diagram similar to that above.

$1 \simeq v \circ w_3$ and $1 \simeq w_3 \circ v$ by homotopies given by:

$$(s(\overline{c}, \omega, a), \sigma, b), t \mapsto (s(\overline{c}, \omega * (\sigma^{bt})^{-1} * (\sigma^{bt}), a + 2bt), \sigma, b).$$

and

$$(s(\overline{c},\omega,a),\sigma,b),t \mapsto (s(\overline{c},\omega*(\sigma_{bt})*(\sigma_{bt})^{-1},a+2bt),\sigma,b).$$

The maps are continuous. I will show the first in detail; the case of the second map is similar.

$$\begin{array}{ccc}
CF\times\Omega B\cup F\times PB\times I & \xrightarrow{\text{the homotopy}} & CF\times\Omega B\cup F\times PB \\
\downarrow & & \downarrow \\
CF\times PB\times I & \longrightarrow & CF\times PB \\
\uparrow & & \uparrow \\
I\times F\times PB\times I & \longrightarrow & I\times F\times PB \\
\downarrow & & \downarrow \\
I\times\overline{C}\times N\times N\times\mathcal{U}(R_+,R_+) & \longrightarrow & I\times\overline{C}\times N\times N
\end{array}$$

So the homotopy is a continuous map if the lower horizontal map is a continuous map. It is

$$1\times m\times 1_N\circ 1\times m\times 1_N\circ 1\times i\times 1_{N\times N}\circ 1\times \mathrm{diag}\times 1_N\circ$$
$$1\times n_2\times 1_N\circ \mathrm{perm}(5,6)\circ 1\times\mathrm{diag}\times 1_{\mathcal{U}(R_+,R_+)}$$

which is continuous. So w_3 is a homotopy equivalence, and $w=w_3\circ w_2\circ w_1$ is a homotopy equivalence. □

Putting this together the map is :

$$w((1-s)(\overline{c},\omega,a)+s(\sigma,b)) =$$
$$(\min(1,2-2s)(\overline{c},\omega*(\sigma^{b\min(1,2s)})^{-1},a+b\min(1,2s)),\sigma^{b\min(1,2s)},b\min(1,2s)).$$

4.3 The Ganea Tower, and the associated spectral sequence

The Ganea Tower

$$
\begin{tikzcd}
 & \vdots & \\
F_{p+1} \arrow[r, "i_{p+1}"] & C_{p+1} \arrow[u, "j_{p+1}"] \arrow[d, bend left, "t_{p+1}"] & \\
 & \overline{C}_{p+1} \arrow[u, "s_{p+1}"] \arrow[r, "k_p"] & SF_p \\
F_p \arrow[r, "i_p"] & C_p \arrow[u, "j_p"] & \\
 & \vdots \arrow[u, "s_p"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \vdots & \\
 & \overline{C}_2 \arrow[u, "s_2"] \arrow[r, "k_1"] & SF_1 \\
F_1 \arrow[r, "i_1"] & C_1 \arrow[u, "j_1"] \arrow[d, bend left, "t_1"] & \\
 & \overline{C}_1 \arrow[u, "s_1"] \arrow[r, "k_0"] & SF_0 \\
F_0 \arrow[r, "i_0"] & C_0 \arrow[u, "j_0"] \arrow[d, bend left, "t_0"] & \\
 & \overline{C}_0 \arrow[u, "s_0"] &
\end{tikzcd}
$$

Spaces and maps are given inductively from $\overline{C}_0 \xrightarrow{\overline{r}} B$. There are also compatible maps $r_p : C_p \to B$ and $\overline{r}_p : \overline{C}_p \to B$.

Given $\overline{C}_i \xrightarrow{\overline{r}_i} B$, all spaces and maps are given as at the beginning of 4.2, except k_i which is the canonical collapse map.

Let G be a linear functor. G-homology G_* is defined by (note the functor is to the based category).

$$G_*(X) = \pi_*(G(X)) \text{ and } G_*(f) = (G(f))_*$$

Lemma 29 *There is a sequence of natural homotopy equivalences connecting $G(X)$ and $\Omega G(SX)$*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & SX \end{tikzcd} \qquad \begin{tikzcd} G(X) \arrow[r, equal] \arrow[d] & G(X) \arrow[r] \arrow[d] & G(*) \arrow[d] & * \arrow[l, "\simeq"'] \arrow[d] \\ * & G(*) \arrow[l, "\simeq"'] \arrow[r] & G(SX) \arrow[r, equal] & G(SX) \end{tikzcd}$$

The first square is cocartesian, hence the middle square in the second diagram is cartesian. By homotopy invariance, the maps between the homotopy fibers are homotopy equivalences (they are clearly natural). $\square$

Corollary 30 *There is a natural suspension isomorphism*

$$G_*(X) \cong \pi_*(\Omega G(SX)) \cong G_{*+1}(SX). \quad \square$$

G_* is a homology theory, so G_* takes cofiber sequences to exact sequences. Hence we have the exact couple:

$$\begin{tikzcd} G_*(C_p) \arrow[rr, "(s_{p+1}\circ j_p)_*"] & & G_*(C_{p+1}) \arrow[dl, "(\text{susp iso})\circ(k_p\circ t_{p+1})_*"] \\ & G_*(F_p) \arrow[ul, "(i_p)_*"] & \end{tikzcd}$$

Here we use the natural suspension isomorphism $G_*(X) \cong G_{*+1}(SX)$. There is an associated spectral sequence with $E^1_{p,q} = G_{p+q}(F_p)$. The d^1-differential is

$$d^1_* \ : \ G_*(F_p) \xrightarrow{(k_{p-1}\circ t_p \circ i_p)_*} G_*(SF_{p-1}) \cong G_{*-1}(F_{p-1}).$$

On the space level the first part is induced by

$$d^1 \ : \ \begin{tikzcd} F_p \arrow[r, "i_p"] & C_p \arrow[d, "t_p"'] & \\ & \overline{C}_p \arrow[r, "k_{p-1}"] & SF_{p-1}. \end{tikzcd}$$

It may not be clear that d^1 is a differential. This is seen by noting that $d^1 \circ d^1$ contains $S(i_p) \circ k_p$. From the Puppe sequence $-S(i_p) \circ k_p$ is null homotopic, hence $S(i_p) \circ k_p = -(-S(i_p) \circ k_p)$ is null homotopic.

Applying G on the pushout square of $(* \leftarrow X \xrightarrow{f} Y)$ gives the following.

Lemma 31 *Given $f : X \to Y$ there is a sequence of natural homotopy equivalences between* hofib $G(f)$ *and* $\Omega G(\text{hocofib}(f))$. $\square$

ΩB is a based space, choose a basepoint of F_{p-1}. Then $CF_{p-1} \cup C\Omega B \overset{\nu}{\hookrightarrow} F_{p-1} * \Omega B$ is a cofibration (since the basepoints are nondegenerated), hence $\text{hocofib}(\nu) \to F_{p-1} * \Omega B/(CF_{p-1} \cup C\Omega B) \cong S^1 \wedge F_{p-1} \wedge \Omega B$ is a homotopy equivalence. Since $CF_{p-1} \cup C\Omega B$ is contractible $F_{p-1} * \Omega B \hookrightarrow \text{hocofib}(\nu)$ is a homotopy equivalence. We have

Lemma 32 $F_p \simeq S^p \wedge F_0 \wedge \Omega B^{\wedge p}$. $\square$

$\mathrm{conn}(X \wedge Y) \geq \mathrm{conn}(X) + \mathrm{conn}(Y) + 1$, so F_p is at least $(p-1)$-connected. So we have a first quadrant spectral sequence. The spectral sequence converges to $G_{p+q+1}(\mathrm{hocofib}(C_0 \to C_\infty)$, where $C_\infty = \mathrm{hocolim}(C_0 \to C_1 \to \cdots \to C_n \to \cdots)$. From the tower we have

$$\begin{tikzcd}
C_\infty \arrow[r] & \mathrm{hocolim}(B = B = \cdots = B = \cdots). \\
C_0 \arrow[u] \arrow[r] & B \arrow[u, "\simeq"'] \\
\overline{C}_0 \arrow[u, "\simeq"] \arrow[ur, "r_0"'] &
\end{tikzcd}$$

The map $C_n \to B$ is n-connected if B is path connected, since the homotopy fiber at the point $b_0 \in B$ is $(p-1)$-connected. B must be path connected since we only know the homotopy fibers at one point.

This gives a sequence of equivalences between $\mathrm{hocofib}(C_0 \to C_\infty)$ and $\mathrm{hocofib}(\overline{C}_0 \to B)$, hence $G_{p+q+1}(\mathrm{hocofib}(C_0 \to C_\infty)) \cong G_{p+q+1}(\mathrm{hocofib}(\overline{C}_0 \to B)) \cong \pi_{p+q}\mathrm{hofib}(G(\overline{C}_0) \to G(B))$.

Summarizing; if B is path connected, we have the spectral sequence

$$E^1_{p,q} = G_q(F_0 \wedge \Omega B^{\wedge p}) \Longrightarrow \pi_{p+q}(\mathrm{hofib}(G(\overline{C}_0) \to G(B))).$$

4.4 The d^1-differential

Because the Ganea map

$$w_p \;:\; F_{p-1} * \Omega B \longrightarrow F_p$$

is a homotopy equivalence, so is

$$(\cdots((w_p * 1_{\Omega B}) * 1_{\Omega B}) * \cdots * 1_{\Omega B})$$

and hence the iteration

$$w^p \equiv w_p \circ (w_{p-1} * 1_{\Omega B}) \circ ((w_{p-2} * 1_{\Omega B}) * 1_{\Omega B}) \circ \cdots \circ (\cdots((w_1 * 1_{\Omega B}) * 1_{\Omega B}) * \cdots * 1_{\Omega B})$$
$$: \; (\cdots((F_0 * \Omega B) * \Omega B) \cdots * \Omega B) \longrightarrow F_p$$

is also a homotopy equivalence.

Let el be an element in $(\cdots((F_0 * \Omega B) * \Omega B) \cdots * \Omega B)$.

$$\mathrm{el} \;=\; (1-t_p)(\cdots((1-t_2)((1-t_1)(\overline{c}, \omega, a) \; + t_1(\sigma_1, a_1)) + t_2(\sigma_2, a_2)) + \cdots) + t_p(\sigma_p, a_p).$$

The following map will be shown to represent the d^1-differential.

$$\delta : \; ((\cdots(F_0 * \overbrace{\Omega B) \cdots) * \Omega B}^{p}) \longrightarrow S((\cdots(F_0 * \overbrace{\Omega B) \cdots) * \Omega B}^{p-1}).$$

It is defined by

$$
\begin{aligned}
\delta(\mathrm{el}) \;=\; & \min(1, 2-2t_p)((1-t_{p-1})(\cdots((1-t_1)(\overline{c}, \omega * \sigma_p^{-1}, a + a_p) + \\
& t_1(\sigma_1 * \sigma_p^{-1}, a_1 + a_p)) + \cdots + t_{p-1}(\sigma_{p-1} * \sigma_p^{-1}, a_{p-1} + a_p)\,))/1(\bullet).
\end{aligned}
$$

The bullet is used for brevity, here it means that we collapse the bottom in the cone.

The map is on the form $X * Y \to S(X \times Y) \to S(Y)$. The first map is continuous by the diagram.

$$
\begin{tikzcd}
X \times Y \times I \arrow[r, two heads] \arrow[dr] & X * Y \arrow[d] \\
& S(X \times Y)
\end{tikzcd}
$$

The second map is continuous if $X \times Y \to Y$ is continuous.

$$
\begin{tikzcd}
F_0 \times I \times \Omega B \times I \times \Omega B \times \cdots \times I \times \Omega B \times \Omega B \arrow[r] \arrow[d] & F_0 \times I \times \Omega B \times I \times \cdots \times I \times \Omega B \arrow[d] \\
((\cdots((F_0 * \Omega B) * \Omega B)\cdots) * \Omega B) \times \Omega B \arrow[r] & ((\cdots((F_0 * \Omega B) * \Omega B)\cdots) * \Omega B)
\end{tikzcd}
$$

The upper left map is continuous by taking p copies of the last map inverted, and acting from the right by path composition on F_0 and on each of the loop spaces.

Theorem 3 *The following diagram is homotopy commutative.*

$$
\begin{tikzcd}
(\cdots((F_0 * \Omega B) * \ \Omega B)\cdots * \Omega B) \arrow[r, "\delta"] \arrow[d, "w^p"'] & S(\cdots((F_0 * \Omega B) * \ \Omega B)\cdots * \Omega B) \arrow[d, "Sw^{p-1}"] \\
F_p \arrow[r, "d^1"] & SF_{p-1}
\end{tikzcd}
$$

I suppress the lengths of the paths in the remainder of the proof.

$$
\begin{aligned}
d^1 \circ w^p(\mathrm{el}) \;=\; & (\min(1, 2-2t_p)(\min(1, 2-2t_{p-1})(\cdots(\min(1, 2-2t_1) \\
& (\overline{c}, \omega * (\sigma_1^{a_1 \min(1,2t_1)})^{-1}), \sigma_1^{a_1 \min(1,2t_1)} * (\sigma_2^{a_2 \min(1,2t_2)})^{-1}, \ldots \\
& \sigma_{p-1}^{a_{p-1} \min(1,2t_{p-1})} * (\sigma_p^{a_p \min(1,2t_p)})^{-1})\cdots)))/1(\bullet)
\end{aligned}
$$

$$
\begin{aligned}
Sw^{p-1} \circ \delta(\mathrm{el}) \;=\; & (\min(1, 2-2t_p)(\min(1, 2-2t_{p-1})(\cdots(\min(1, 2-2t_1) \\
& (\overline{c}, \omega * \sigma_p^{-1} * ((\sigma_1 * \sigma_p^{-1})^{(a_1+a_p)\min(1,2t_1)})^{-1}, \cdots (\sigma_{p-2} * \sigma_p^{-1})^{(a_{p-2}+a_p)\min(1,2t_{p-2})} \\
& *(\sigma_{p-1} * \sigma_p^{-1})^{(a_{p-1}+a_p)\min(1,2t_{p-1})})^{-1}, (\sigma_{p-1} * \sigma_p^{-1})^{(a_{p-1}+a_p)\min(1,2t_{p-1})}))\cdots))/1(\bullet).
\end{aligned}
$$

We will now construct two homotopies $H^1 : \ Sw^{p-1} \circ \delta \simeq H_1^1$ and $H^2 : d^1 \circ w^p \simeq H_1^1$, which will prove the theorem.

The first homotopy is

$$
H^1 : ((\cdots(F_0 * \Omega B) * \cdots) * \Omega B) \times I \to SF_{p-1}
$$

$$
\begin{aligned}
H^1_s(\mathrm{el}) = {} & (\min(1,2-2t_p)(\min(1,2-2t_{p-1})(\cdots(\min(1,2-2t_1) \\
& (\bar{c}, \omega * \sigma_p^{-1} * ((\sigma_1 * \sigma_p^{-1})^{(a_1+a_p)\min(1,2t_1)+sa_p(1-\min(1,2t_1))})^{-1}), \cdots \\
& (\sigma_{p-2} * \sigma_p^{-1})^{(a_{p-2}+a_p)\min(1,2t_{p-2})+sa_p(1-\min(1,2t_{p-2}))} \\
& *((\sigma_{p-1} * \sigma_p^{-1})^{(a_{p-1}+a_p)\min(1,2t_{p-1})+sa_p(1-\min(1,2t_{p-1}))})^{-1}, \\
& (\sigma_{p-1} * \sigma_p^{-1})^{(a_{p-1}+a_p)\min(1,2t_{p-1})+sa_p(1-\min(1,2t_{p-1}))})\ldots))/1(\bullet)
\end{aligned}
$$

We have the diagram

$$
\begin{tikzcd}
(\cdots(F_0 * \Omega B) * \cdots * \Omega B) \times I \arrow[rr, "H^1"] & & SF_{p-1} \\
F_0 \times I \times \Omega B \times \cdots \times I \times \Omega B \times I \arrow[u, two heads] \arrow[rr] \arrow[dr, "\text{map } 2"'] & & I \times (\mathrm{coll}^{-1}(F_{p-1})) \arrow[u, two heads, "\text{map } 1"'] \arrow[d, hook] \\
& & I \times C_0 \times N \times \overbrace{N \times I \times \cdots \times N \times I}^{p-1}
\end{tikzcd}
$$

The space $\mathrm{coll}^{-1}(F_{p-1})$, where coll means the collapse map, is given by the diagram.

$$
\begin{tikzcd}
F_{p-1} \arrow[r, hook, "\text{closed}"] & C(C\cdots C(CF_0 \cup C_0 \times N) \cup C_1 \times N \cdots) \cup C_{p-1} \times N \\
(\mathrm{coll})^{-1}(F_{p-1}) \arrow[u, two heads] \arrow[r, hook] & I(\cdots(I \times (I \times F_0 \cup C_0 \times N) \cup C_1 \times N) \cup C_2 \cdots) \cup C_{p-1} \times N \arrow[u, two heads] \arrow[d, hook] \\
& I^p \times C_0 \times \overbrace{N \times \cdots \times N}^{p}
\end{tikzcd}
$$

Here closed means closed subset, this follows by induction and that F_q is a closed subset of $CF_{q-1} \cup C_{q-1} \times N$. By Lemma 15, F_p has the coinduced topology from $(\mathrm{coll})^{-1}(F_p)$, which in turn gets the topology from the inclusion. map 2 consists of many compositions of continuous maps: diagonal, permutations, i, m, n_2 and maps into R_+ of typical form $a, t_1, t_2, t_3 \mapsto (a(\min(1,2t_1)) + t_2t_3(1-\min(1,2t_1)))$, which are all continuous. So H^1 is continuous.

$$
\begin{aligned}
H^1_1(\mathrm{el}) \;=\; & \min(1,2-2t_p)(\cdots\min(1,2-2t_1)(\bar{c}, \omega * \sigma_p^{-1} * \sigma_p * (\sigma_1^{a_1\min(1,2t_1)})^{-1}, \ldots \\
& \sigma_{p-2}^{a_{p-2}\min(1,2t_{p-2})} * \sigma_p^{-1} * \sigma_p * (\sigma_{p-1}^{a_{p-1}\min(1,2t_{p-1})})^{-1}, \sigma_{p-1}^{a_{p-1}\min(1,2t_{p-1})} * \sigma_p^{-1}))/1(\bullet)
\end{aligned}
$$

The second homotopy is

$$
\begin{aligned}
H_s^2(\mathrm{el}) = (\min(1, 2-2t_p)(\min(1, 2-2t_{p-1})(\cdots \min(1, 2-2t_1) \\
(\overline{c}, \omega * (\sigma_p^{a_p s})^{-1} * \sigma_p^{a_p s} * (\sigma_1^{a_1 \min(1,2t_1)})^{-1}), \ldots \sigma_{p-2}^{a_{p-2}\min(1,2t_{p-2})} * \\
(\sigma_p^{a_p s})^{-1} * \sigma_p^{a_p s} * (\sigma_{p-1}^{a_1 \min(1,2t_{p-1})})^{-1}, \sigma_{p-1}^{a_{p-1}\min(1,2t_{p-1})} * \sigma_p^{-1}) \cdots))/1(\bullet).
\end{aligned}
$$

$H_1^2 \;=\; H_1^1$ and

$$
\begin{aligned}
H_0^2(\mathrm{el}) \;&=\; \min(1, 2-2t_p)(\cdots \min(1, 2-2t_1)(\overline{c}, \omega * (\sigma_1^{a_1 \min(1,2t_1)})^{-1}, \ldots \\
&\sigma_{p-2}^{a_{p-2}\min(1,2t_{p-2})} * (\sigma_{p-1}^{a_{p-1}\min(1,2t_{p-1})})^{-1}, \sigma_{p-1}^{a_{p-1}\min(1,2t_{p-1})} * \sigma_p^{-1}))/1(\bullet)
\end{aligned}
$$

This is $d^1 \circ w^p$ since $\min(1, 2t_p) < 1$ only when $\min(1, 2-2t_p) = 1$, and then it is collapsed. Note that $(\sigma_p^{a_p}, a_p) = (\sigma_p, a_p)$.

H^2 is continuous by the same argument as for continuity of H^1.

$H^1 : \; Sw^{p-1} \circ \delta \simeq H_1^1$ and $H^2 : \; d^1 \circ w^p \simeq H_1^1$ hence $Sw^{p-1} \circ \delta \simeq d^1 \circ w^p$. $\square$

For B path connected the spectral sequence is

$$E^1_{p,q} = G_{p+q}(F_0 * \Omega B^{*p}) \Rightarrow \pi_{p+q}(\mathrm{hofib}(G(\overline{C}_0) \to G(B)))$$

where the d^1-differential is the suspension isomorphism composed with $G_*(\delta)$.

We might consider δ as a Hopf construction. Given a map $m : X \times Y \longrightarrow Y$ the Hopf construction $H(m) : X * Y \longrightarrow SY$ is defined by $(1-s)x + sy \mapsto [m(x, y), s]$.

Let the left action of ΩB on $((\cdots(F_0 * \overbrace{\Omega B) \cdots) * \Omega B}^{p})$ be given by :

$$
\begin{aligned}
&((\sigma, \overline{a}), \mathrm{el}) \longmapsto \\
(1-t_p)(\cdots((1-t_1)(\overline{c}, \omega * \sigma^{-1}, a + \overline{a}) \; &+ t_1(\sigma_1 * \sigma^{-1}, a_1 + \overline{a})) + \\
t_2(\sigma_2 * \sigma^{-1}, a_2 + \overline{a}) + \cdots) &+ t_p(\sigma_p * \sigma^{-1}, a_p + \overline{a}))
\end{aligned}
$$

Denote this map by m; it is a monoid action. Let 1_{SX} and $2_{SX} : SX \longrightarrow SX$ respectively be given by $[s, x] \mapsto [s, x]$ and $[s, x] \mapsto [\min(1, 2s), x]$. They are homotopic.

Corollary 33 $\quad \delta \;=\; 2_{SX} \circ H(m) \simeq H(m). \quad \square$

Examples

Let $B \in \mathcal{W}$ be a path connected space. Given a map $C \xrightarrow{f} B$, let $F = \mathrm{hofib}_{b_0}(f)$.

- Let H be (reduced) singular homology, with field coefficients. By the Künneth theorem we have the spectral sequence

$$E^1_{p,q} = (H(F) \otimes H(\Omega B)^{\otimes p})_q \Rightarrow H_{p+q+1}(\text{hocofib}(f)).$$

I have not been able to describe the d^1-differential here. Note that homology theories in general do not satisfy a Kunneth theorem.

- From [Stash] we have that

$$C \simeq B(*, \Omega B, F).$$

The map $f : C \to B$ is equivalent to

$$B(*, \Omega B, F) \to B(*, \Omega B, *)$$

The space of q-simplices is $B_q(*, \Omega B, F) = \Omega B^{\times p} \times F$.

$$\text{hocofib}(f) \simeq \text{hocofib}(B(*, \Omega B, F) \to B(*, \Omega B, *)) =$$

$$(C(\amalg_{q \geq 0} \Delta^q \times \Omega B^{\times q} \times F/\sim) \amalg (\amalg_{q\geq 0} \Delta^q \times \Omega B^{\times q})/\sim)/\sim$$

with $1 \times \Delta^q \times \Omega B^{\times q} \times f' \sim \Delta^q \times \Omega B^{\times q}$ for all $f' \in F$. The relation inside the parenthesis is given in Appendix 1. This equals

$$(\amalg_{q\geq 0} \Delta^q \times \Omega B^{\times q} \times SF/\sim)/(\amalg_{q \geq 0} \Delta^q \times \Omega B^{\times q} \times 0/\sim)$$

$$\text{hocofib}(f) \simeq B(*, \Omega B, SF)/B(*, \Omega B, *)$$

From Lemma 31 we obtain

$$\text{hofib}\, G(f) \simeq \Omega G(B(*, \Omega B, SF)/B(*, \Omega B, *)).$$

The bar filtration gives the following p-th subquotient

$$S^p \wedge \Omega B^{\wedge p} \wedge SF.$$

So the spectral sequence induced from the bar filtration is

$$E^1_{p,q} = G_{q-1}(\Omega B^{\wedge p} \wedge F) \Rightarrow G_{p+q}(\text{hocofib}(f)).$$

This spectral sequence has equal terms as the Ganea spectral sequence (up to grading). I do not know if they are equal as spectral sequences.

- In the case $* \xrightarrow{b_0} B$, $F = \Omega_{b_0} B$, we get a delooping spectral sequence

$$E^1_{p,q} = G_{p+q}(\Omega B^{*p+1}) \Rightarrow G_{p+q+1}(B).$$

4.5 Extension to spaces over X

Let $\mathcal{G}$Mod be the category of graded $\mathbb{Z}$-modules. The morphism sets are given the discrete topology.

A homology theory over X is a covariant functor

$$G_* : \mathcal{U}_X \longrightarrow \mathcal{G}\mathrm{Mod}$$

satisfying the following three axioms

- The homotopy axiom: If $\gamma : f \to g$ is a homotopy equivalence, that is if $\gamma : D(f) \to D(g)$ is a homotopy equivalence, then $G_*(\gamma)$ is an isomorphism.
- The suspension axiom (excision): $G_{*+1} \circ S_X \cong G_*$.
- Long exact sequence: G_* takes cofiber sequences (in $\mathcal{U}_X$) to exact sequences.

It is easy to see that the homotopy axiom is equivalent to the following statement: If $\gamma, \gamma' : f \to g$ are homotopic maps then $G_*(\gamma) = G_*(\gamma')$. The requirement that G_* is a continuous functor would imply the homotopy axiom.

A linear functor $G : \mathcal{U}_X \to \mathcal{T}$ gives rise to a homology theory by

$$G_* = \pi_* \circ G$$

(I do not know if all homology theories are of this form.)

An example of monoid action on a homology theory: Let F be a stably excisive homotopy functor. We then saw that the linear functor $G = P\widetilde{F}$ had an $\Omega F(1_X)$-action, $\Omega F(1_X) \to \mathcal{E}_X(G, G)$. The homotopy group functor gives

$$\pi_* : \mathcal{E}_X(G, G) \to \mathcal{G}\mathrm{Mod}(G_*, G_*)$$

giving a monoid action $\Omega F(1_X) \times G_* \to G_*$. π is a continuous functor, so the action then factors through the path components of $\Omega F(1_X)$, which are $\pi_1(F(1_X))$.

The homotopy cofiber in $\mathcal{U}_X$ is defined by

$$\mathrm{hocofib}(f \xrightarrow{\gamma} g) = \mathrm{hocolim}(1_X \leftarrow f \to g).$$

G takes a cofiber sequence in $\mathcal{U}_X$ to a fiber sequence in $\mathcal{T}$. For simplicity we here also write elements in $\mathcal{U}_X$ by their underlying spaces (their domains).

Suppose given

$$\begin{tikzcd} \overline{D}_0 \arrow[rr, "\overline{p}_0"] \arrow[dr, "\overline{d}_0"'] & & B \arrow[dl, "b"] \\ & X & \end{tikzcd}$$

an inclusion $r : X \to B$, and a point $b_0 \in B$ such that $b \circ r = 1_X$ and $r \circ b(b_0) = b_0$. Define spaces and maps inductively by:

$$D_p = \{\, (d, \omega, a) \in \overline{D}_p \times N \mid \overline{\rho}_p(d) = \omega(0) \,\}$$

$$\rho_p(d, \omega, a) = \omega(a)$$

$$F_p = \rho_p^{-1}(b_0)$$

$$\overline{D}_{p+1} = \mathrm{hocofib}(F_p \to D_p).$$

The map ρ_{p+1} to B is induced by
$$\begin{tikzcd} X \arrow[d, "r"] & F_p \arrow[l, "\text{to } x"'] \arrow[r] \arrow[d, "\text{to } b_0"] & D_p \arrow[d] \\ B \arrow[r, equal] & B \arrow[r, equal] & B. \end{tikzcd}$$

The maps to X is compositions of the maps to B by the map b. Using an exact couple as before, with G-homology, we get a spectral sequence with

$$E^1_{p,q} = G_{p+q}(F_p)$$

(we here make use of the Puppe cofibration sequence in $\mathcal{U}_X$) which possibly converges to $G_{p+q+1}(\mathrm{hocofib}(D_0 \to D_\infty))$. The inductive step is

$$\begin{tikzcd} & \overline{D}_{p+1} \arrow[r, "\overline{\rho}_{p+1}"] & B \arrow[d, equal] \\ F_p \arrow[r] & D_p \arrow[u, "j_p"] \arrow[r, "\rho_p"] & B \arrow[d, equal] \\ & \overline{D}_p \arrow[u, "s_p"] \arrow[r, "\overline{\rho}_p"] & B \end{tikzcd}$$

s_p and j_p are maps in $\mathcal{U}_X$.

Extending the Ganea map:

$$\mathrm{hofib}_{b_0}(r) \cup (F_{p-1} * \Omega B) \to \mathrm{hofib}_{b_0}(r) \cup (CF_{p-1} \times \Omega B) \cup F_{p-1} \times C\Omega B \to$$

$$\mathrm{hofib}_{b_0}(r) \cup (CF_{p-1} \times \Omega B) \cup F_{p-1} \times PB \to \mathrm{hofib}_{b_0}(r) \cup (CF_{p-1} \times \Omega B) \cup$$
$$\mathrm{hofib}_{b_0}(D_p \to B) = \mathrm{hofib}_{b_0}(\overline{D}_p \to B) = F_{p+1}.$$

These maps are over B, and they are equivalences.

In the case when $X = B$, $b = r = 1_B$ and b_0 is an arbitrary point in B we have $F_p \simeq PB \cup (F_{p-1} * \Omega B) \xrightarrow{\simeq} S^1 \wedge (F_{p-1})_+ \wedge \Omega B$ and it follows that F_p is $(\mathrm{conn}(\Omega B) + 1)$-connected.

In this spectral sequence the fibers behave badly, they are also very special objects in $\mathcal{U}_X$, they are point maps.

Another tower. We use the same notation as above. Given

$$\begin{tikzcd} \overline{D}_0 \arrow[r, "\overline{\rho}_0"] \arrow[d, "\chi"'] & B \\ X & \end{tikzcd}$$

Define spaces and maps inductively by:

$$D_p = \{ (d, \omega, a) \in \overline{D}_p \times N \mid \overline{\rho}_p(d) = \omega(0) \}$$

the map to X is: $(d, \omega, a) \mapsto \chi(d)$.

$$\rho_p(d, \omega, a) = \omega(a) \text{ and } F_p = \rho_p^{-1}(b_0)$$

$$\overline{D}_{p+1} = \operatorname{hocofib}(F_p \to D_p).$$

The maps to X and B are defined by

$$\begin{tikzcd} X \arrow[d, equal] & D_p \arrow[l] \arrow[r] & B \arrow[d, equal] \\ X \arrow[d, equal] & F_p \arrow[l] \arrow[u] \arrow[r, "\text{to } b_0"] \arrow[d] & B \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r, "\text{to } b_0"] & B. \end{tikzcd}$$

Extending the Ganea map:

$$C_X F_{p-1} \times \Omega B \cup F_{p-1} \times C\Omega B \longrightarrow \operatorname{hofib}(\overline{D}_p \to B) = F_p.$$

This map over X is an equivalence. We get a spectral sequence as above. I suppose that $\operatorname{conn}(F_p \to X)$ approaches infinity with p. I have not been able to use these two spectral sequences.

I also tried to use homotopy fibers in the category of retractive spaces over X. The inclusions of points is then replaced by the inclusion of X. The spectral sequence associated to this tower, gives no information. This could not have been used in the chain rule problem, since the constructions only coincide with those in our problem when X is a point.

5 Appendix 1

5.1 Topological categories

A topological category $\mathcal{C}$ is a small category, with topologized morphism sets and continuous compositions. The identity elements are required to be closed nondegenerated basepoints in the corresponding morphism spaces. The object category of $\mathcal{C}$, denoted id$\mathcal{C}$ has the same objects, but only the identity morphisms. We may state the last requirement by; $\mathrm{mor}(\mathrm{id}\mathcal{C}) \hookrightarrow \mathrm{mor}(\mathcal{C})$ is a closed cofibration. In what follows $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ will denote topological categories. A continuous covariant (contravariant) functor $X : \mathcal{C} \to \mathcal{U}$ is called a left (right) $\mathcal{C}$-module, denoted ${}_{\mathcal{C}}X, (X_{\mathcal{C}})$, in particular we have ${}_{\mathcal{C}}X = X_{\mathcal{C}^{\mathrm{op}}}$. A $\mathcal{C}$-$\mathcal{D}$-bimodule is a continuous (covariant) functor $X : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{U}$, denoted ${}_{\mathcal{C}}X_{\mathcal{D}}$. All functors between topological categories are required to be continuous. Let ${}_{\mathcal{C}}\mathrm{Mod}_{\mathcal{D}}$ be the functor category of $\mathcal{C}$-$\mathcal{D}$-bimodules. A map $\gamma : X \to Y$ is an $\mathcal{C}$-equivalence if it is a homotopy equivalence in the category of $\mathcal{C}$-modules, that is, there is a homotopy inverse in the category, and the homotopies are compatible. It is an id$\mathcal{C}$-equivalence if each $\gamma(C)$ is a homotopy equivalence. γ is a homeomorphism if each $\gamma(C)$ is a homeomorphism. Homeomorphisms are denoted by $\cong$.

Let $*$ be the one object and one morphism category. ${}_{*}\mathrm{Mod}_{\mathcal{C}}$ is denoted by $\mathrm{Mod}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathrm{Mod}_{*}$ is denoted by ${}_{\mathcal{C}}\mathrm{Mod}$, note that $\mathrm{Mod}_{*} = \mathcal{U}$. The morphism set ${}_{\mathcal{C}}\mathrm{Mod}_{\mathcal{D}}(X, Y)$ is given the subspace topology from

$$\prod_{A \in \mathcal{C}, B \in \mathcal{D}} \mathcal{U}(X(A, B), Y(A, B)).$$

For X a left $\mathcal{C}$-module, $f : A \to B$ in $\mathcal{C}$, and $a \in X(A)$, $X(f)a$ is denoted $f \cdot a$. For Y a right $\mathcal{C}$-module, and $b \in Y(B)$, $Y(f)b$ is denoted $b \cdot f$.

The tensor product of a right $\mathcal{C}$-module X, and a left $\mathcal{C}$-module Y is

$$X \otimes_{\mathcal{C}} Y = \amalg_{C \in \mathcal{C}} X(C) \times Y(C) / \sim$$

where $\sim$ is generated by $(x \cdot f, y) \sim (x, f \cdot y)$ for all $f \in \mathrm{mor}(\mathcal{C})$ making this well defined.

Let $\mathcal{C}$ also denote the $\mathcal{C}$-$\mathcal{C}$-bimodule, defined by $C, C' \mapsto \mathcal{U}(C, C')$.

Proposition 5

Given modules ${}_{\mathcal{A}}X_{\mathcal{B}}, {}_{\mathcal{B}}Y_{\mathcal{C}}, {}_{\mathcal{C}}Z_{\mathcal{D}}, {}_{\mathcal{A}}U_{\mathcal{D}}$, and ${}_{\mathcal{A}}V_{\mathcal{C}}$.

1. $X \otimes_{\mathcal{B}} (Y \otimes_{\mathcal{C}} Z) \cong (X \otimes_{\mathcal{B}} Y) \otimes_{\mathcal{C}} Z$ as $\mathcal{A}$-$\mathcal{D}$-bimodules.

2. $\mathcal{A} \otimes_{\mathcal{A}} X \cong X \cong {}_{\mathcal{A}}\mathrm{Mod}(\mathcal{A}, X)$ as $\mathcal{A}$-$\mathcal{B}$-bimodules.

3. $X \otimes_{\mathcal{B}} Y \cong Y \otimes_{\mathcal{B}^{op}} X$ as $\mathcal{A}$-$\mathcal{C}$-bimodules.

4. ${}_{\mathcal{A}}\mathrm{Mod}(X \otimes_{\mathcal{B}} Y, U) \cong {}_{\mathcal{B}}\mathrm{Mod}(Y, {}_{\mathcal{A}}\mathrm{Mod}(X, U))$ as $\mathcal{C}$-$\mathcal{D}$-bimodules.

5. ${}_{\mathcal{A}}\mathrm{Mod}_{\mathcal{C}}(X \otimes_{\mathcal{B}} Y, V) \cong {}_{\mathcal{A}}\mathrm{Mod}_{\mathcal{B}}(X, \mathrm{Mod}_{\mathcal{C}}(Y, V))$.

A functor $F : \mathcal{C} \to \mathcal{D}$ induces a forgetful functor $F^* : {}_{\mathcal{D}}\mathrm{Mod} \to {}_{\mathcal{C}}\mathrm{Mod}$, by $X \mapsto X \circ F$. The left Kan extension of X along F is $\mathcal{D}_{\mathcal{C}} \otimes_{\mathcal{C}} X$. The functor

$$ {}_{\mathcal{C}}\mathrm{Mod} \to {}_{\mathcal{D}}\mathrm{Mod},\ X \mapsto \mathcal{D}_{\mathcal{C}} \otimes_{\mathcal{C}} X $$

is a left adjoint of F^*. The right Kan extension of X along F is ${}_{\mathcal{C}}\mathrm{Mod}({}_{\mathcal{C}}\mathcal{D}, X)$. The functor

$$ {}_{\mathcal{C}}\mathrm{Mod} \to {}_{\mathcal{D}}\mathrm{Mod},\ X \mapsto {}_{\mathcal{C}}\mathrm{Mod}({}_{\mathcal{C}}\mathcal{D}, X) $$

is a right adjoint of F^*. We will often omit writing the forgetful functor F^*. The functor F^* is analogous to restriction of scalars, and its left adjoint is analogous to extension of scalars. We also have that $F^*X \cong {}_{\mathcal{C}}\mathcal{D} \otimes_{\mathcal{D}} X$ and $F^*X \cong {}_{\mathcal{D}}\mathrm{Mod}(\mathcal{D}_{\mathcal{C}}, X)$.

Let Δ denote a skeleton of the category of total ordered finite sets and order-preserving functions; with objects $[n] = \{0, 1, \ldots, n\}$, and maps generated by $s_i : [n] \to [n-1]$, defined by mapping j to j for $j \le i$ and j to $j-1$ for $j > i$, and $d_i : [n-1] \to [n]$, defined by mapping j to j for $j < i$ and by mapping j to $j+1$ for $j \ge i$. A right Δ module is called a simplicial space. If X is a simplicial space, $X([n])$ is denoted X_n, and is called the space of n-simplices. $X(s_i)$ is denoted s_i and is called the i-th degeneration map, and $X(d_i)$ is denoted d_i, and is called the i-th face map.

The simplex functor ∇ is a left Δ module. $\nabla(n)$ is the n-th standard simplex $\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,|\, t_0 + \cdots + t_n = 1 \,\mathrm{and}\, t_i \ge 0\}$.

The face maps are $d_i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n)$, and the degeneration maps are $s_i(t_0, \ldots, t_n) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_n)$.

The topological realization of a simplical space X is $X \otimes_\Delta \nabla$.

The singular simplex functor $\mathcal{U} \to \mathrm{Mod}_\Delta$ is given by $X \mapsto \mathcal{U}(\nabla, X)$. From Proposition 5.5 it is right adjoint to the topological realization functor.

Limits and colimits over a continuous indexing category $\mathcal{C}$ are in this language given as

$$ * \otimes_{\mathcal{C}} X = \mathrm{colim}_{\mathcal{C}} X $$

$$ {}_{\mathcal{C}}\mathrm{Mod}(*, X) = \lim_{\mathcal{C}} X. $$

Let M be a topological monoid, with closed nondegenerated identity. Let M be the one object category with M as morphism sets. A left M-module is an M-space.

5.2 The bar construction

The bar construction is a homotopy invariant version of the tensor product. It is the topological realization of the bar simplicial space $B_\bullet(X, \mathcal{C}, Y)$. Given modules $X_{\mathcal{C}}$ and ${}_{\mathcal{C}}Y$ it is defined by

$$ B_n(X, \mathcal{C}, Y) = \{(y; f_n, \ldots f_1; x) \,|\, f_n \circ \cdots \circ f_1 : A \to B \text{ defined},\ x \in X(A), y \in Y(B)\}. $$

Face and degeneration maps are given by

$$d^i(y; f_n, \dots f_1; x) = \begin{cases} (y; f_n, \dots f_2; f_1 \cdot x) & i = 0 \\ (y; f_n, \dots, f_{i+1} \circ f_i, \dots, f_1; x) & 0 < i < n \\ (y \cdot f_n; f_{n-1}, \dots f_1; x) & i = n \end{cases}$$

$$s^i(y; f_n, \dots f_1; x) = (y; f_n, \dots, f_{i+1}, 1, f_i, \cdots, f_1; x)$$

With bimodules ${}_{\mathcal{C}}X_{\mathcal{D}}$ and ${}_{\mathcal{A}}Y_{\mathcal{C}}$, the bar construction is seen to be an $\mathcal{A}$-$\mathcal{D}$-bimodule.

$B_\bullet(*, \mathcal{C}, *)$ is called the nerve of the category $\mathcal{C}$. The topological realization of the nerve, $B(*, \mathcal{C}, *)$ is the classifying space of the category, denoted $B\mathcal{C}$. If M is a grouplike topological monoid then ΩBM is homotopy equivalent to M as an H-space. Grouplike means that $\pi_0(M)$ is a group under the monoid multiplication. For arbitrary monoids there are reasons to regard ΩBM as a (homotopy) group completion of M.

Given modules ${}_{\mathcal{C}}X, {}_{\mathcal{D}}X', Y_{\mathcal{C}}$ and $Y'_{\mathcal{D}}$, a functor $F : \mathcal{C} \to \mathcal{D}$, and morphisms $\alpha : X \to F^*X'$ and $\beta : Y \to F^*Y'$, they induce a map

$$B(\beta, F, \alpha) : B(Y, \mathcal{C}, X) \to B(Y', \mathcal{D}, X').$$

Proposition 6

Given modules ${}_{\mathcal{C}}X_{\mathcal{D}}, {}_{\mathcal{A}}Y_{\mathcal{C}}, {}_{\mathcal{D}}Z_{\mathcal{E}}, {}_{\mathcal{E}}E_{\mathcal{F}}$, and ${}_{\mathcal{G}}V_{\mathcal{E}}$.

1. $B(Y, \mathcal{C}, X) \otimes_{\mathcal{D}} Z \cong B(Y, \mathcal{C}, X \otimes_{\mathcal{D}} Z)$ as $\mathcal{A}$-$\mathcal{E}$-bimodules.

2. $B(B(Y, \mathcal{C}, X) \otimes_{\mathcal{D}} Z, \mathcal{E}, U) \cong B(Y, \mathcal{C}, X) \otimes_{\mathcal{D}} B(Z, \mathcal{E}, U) \cong B(Y, \mathcal{C}, X \otimes_{\mathcal{D}} B(Z, \mathcal{E}, U))$ as $\mathcal{A}$-$\mathcal{F}$-bimodules.

3. $B(B(Y, \mathcal{C}, X), \mathcal{D}, Z) \cong B(Y, \mathcal{C}, B(X, \mathcal{D}, Z))$ are equivalent as $\mathcal{A}$-$\mathcal{E}$-bimodules.

4. $B(Y, \mathcal{C}, X) \times B(V, \mathcal{E}, U) \cong B(Y \times V, \mathcal{C} \times \mathcal{E}, X \times U)$ as $\mathcal{A}, \mathcal{G}$-$\mathcal{D}, \mathcal{F}$-multimodules.

5. The evaluation map $B(\mathcal{C}, \mathcal{C}, X) \to X$ is an id$\mathcal{C}$-$\mathcal{D}$-equivalence.

6. Given a functor $F : \mathcal{C} \to \mathcal{D}$ which is identity on objects and homotopy equivalences on morphism spaces. Modules ${}_{\mathcal{C}}X_{\mathcal{E}}, {}_{\mathcal{A}}Y_{\mathcal{C}}, {}_{\mathcal{D}}X'_{\mathcal{E}}$ and ${}_{\mathcal{A}}Y'_{\mathcal{D}}$ an id$\mathcal{C}$-$\mathcal{E}$-equivalence $\alpha : X \to X'$ and an $\mathcal{A}$-id$\mathcal{C}$-equivalence $\beta : Y \to Y'$, then $B(\beta, F, \alpha) : B(Y, \mathcal{C}, X) \to B(Y', \mathcal{D}, X')$ is an $\mathcal{A}$-$\mathcal{E}$-equivalence.

7. Given a map $\tau : F \to G$ of functors $\mathcal{C} \to \mathcal{D}$, modules ${}_{\mathcal{C}}X, {}_{\mathcal{D}}X', Y_{\mathcal{C}}$, and $Y'_{\mathcal{C}}$, and morphisms $\alpha : X \to F^*X'$, $\beta : Y \to F^*Y'$, $\gamma : X \to G^*X'$, and $\delta : Y \to G^*Y'$. Then the induced maps are homotopic.

$$B(\beta, F, \alpha), B(\delta, G, \gamma) : B(Y, \mathcal{C}, X) \longrightarrow B(Y', \mathcal{D}, X').$$

5.3 Homotopy limits

First we consider over and under categories, and more generally the comma category. Suppose given categories and functors

$$\mathcal{D} \xrightarrow{D} \mathcal{C} \xleftarrow{E} \mathcal{E}.$$

The comma category, denoted (D, E) has as objects (d, f, e), where d is an object in $\mathcal{D}$, e is an object in $\mathcal{E}$, and $f : D(d) \to E(e)$. Morphisms between (d, f, e) and (d', f', e') are tuples (g, h), where $g : d \to d'$ and $h : e \to e'$ are maps so that $f' \circ D(g) = E(h) \circ f$.

The over category $(F \downarrow D)$ for an object D in $\mathcal{D}$ is the comma category of $\mathcal{C} \xrightarrow{F} \mathcal{D} \hookleftarrow D$. The under category $(D \downarrow F)$ is the comma category of $D \hookrightarrow \mathcal{D} \xleftarrow{F} \mathcal{C}$. The correspondence $D \mapsto (D \downarrow F)$ defines a continuous contravariant functor from $\mathcal{D}$ to the category of topological categories. Composition with the classifying functor B gives a right $\mathcal{D}$-module, denoted $B(\mathcal{D} \downarrow F)$. $D \mapsto (F \downarrow D)$ defines in the same way a left $\mathcal{D}$-module, denoted $B(F \downarrow \mathcal{D})$.

Given a diagram in $\mathcal{U}$, it is a continuous functor $X : \mathcal{C} \to \mathcal{U}$, with $\mathcal{C}$ the (topological) indexing category. The homotopy colimit of F is defined to be

$$\mathrm{hocolim}_{\mathcal{C}} X = B(*, \mathcal{C}, X).$$

Equivalent descriptions are

$$B(*, \mathcal{C}, X) \cong B(*, \mathcal{C}, \mathcal{C}) \otimes_{\mathcal{C}} X \cong B(\mathcal{C} \downarrow 1_{\mathcal{C}}) \otimes_{\mathcal{C}} X.$$

We may also interpret the homotopy colimit as the colimit of a resolution of the diagram

$$B(*, \mathcal{C}, X) \cong * \otimes_{\mathcal{C}} B(\mathcal{C}, \mathcal{C}, X) = \mathrm{colim}_{\mathcal{C}} B(\mathcal{C}, \mathcal{C}, X).$$

The homotopy limit is defined by

$$\begin{aligned} \mathrm{holim}_{\mathcal{C}} X &= {}_{\mathcal{C}}\mathrm{Mod}(B(1_{\mathcal{C}} \downarrow \mathcal{C}), X) \\ &\cong {}_{\mathcal{C}}\mathrm{Mod}(B(\mathcal{C}, \mathcal{C}, *), X) \cong {}_{\mathcal{C}}\mathrm{Mod}(B(\mathcal{C}, \mathcal{C}, \mathcal{C}) \otimes_{\mathcal{C}} *, X) \\ &\cong {}_{\mathcal{C}}\mathrm{Mod}(*, {}_{\mathcal{C}}\mathrm{Mod}(B(\mathcal{C}, \mathcal{C}, \mathcal{C}), X)) = \lim_{\mathcal{C}}({}_{\mathcal{C}}\mathrm{Mod}(B(\mathcal{C}, \mathcal{C}, \mathcal{C}), X)) \end{aligned}$$

This interprets homotopy limit as a limit of a resolution of the $\mathcal{C}$-diagram X.

$F : \mathcal{C} \to \mathcal{D}$ is called right cofinal if $B(D \downarrow F) \simeq *$ for all $D \in \mathcal{D}$, this is equivalent to that $B(\mathcal{D} \downarrow F) \cong B(*, \mathcal{C}, \mathcal{D}) \to *$ is an id-$\mathcal{D}$-equivalence. Examples of this is: If F is inclusion of the $*$ category to the terminal object in $\mathcal{C}$ (if any). If $\mathcal{C}$ is $(\mathbb{N}, \geq)$, the inclusion of the full subcategory with objects $j \geq n$ for some n.

F is left cofinal if $B(F \downarrow D) \simeq *$ for all $D \in \mathcal{D}$, this is equivalent to that $B(F \downarrow \mathcal{D}) \cong B(\mathcal{D}, \mathcal{C}, *) \to *$ is an id-$\mathcal{D}$-equivalence. An example of this is the inclusion of an initial object (if any).

Proposition 7

Given a functor $F : \mathcal{C} \to \mathcal{D}$ and modules ${}_{\mathcal{C}}X, {}_{\mathcal{C}}Y$ and ${}_{\mathcal{D}}Z$.

Homotopy invariance. If $\gamma : X \to Y$ is an $\mathrm{id}_{\mathcal{C}}$-equivalence, then $\mathrm{holim}(X) \xrightarrow{\simeq} \mathrm{holim}(Y)$ and $\mathrm{hocolim}(X) \xrightarrow{\simeq} \mathrm{hocolim}(Y)$.

Weak invariance. If $\gamma : X \to Y$ is an n-equivalence (each $\gamma(C)$ is an n-equivalence, $C \in \mathcal{C}$) then $\mathrm{hocolim}(X) \to \mathrm{hocolim}(Y)$ is an n-equivalence.

If γ is an equivalence (∞-equivalence), then $\mathrm{holim}(X) \to \mathrm{holim}(Y)$ is an equivalence.

Cofinality theorems. If F is right cofinal, then $\mathrm{hocolim}_{\mathcal{C}} F^* Z \to \mathrm{hocolim}_{\mathcal{D}} Z$ is a homotopy equivalence.

If F is left cofinal then $\mathrm{holim}_{\mathcal{D}} Z \to \mathrm{holim}_{\mathcal{C}} F^* Z$ is a homotopy equivalence.

Reduction theorem. $\mathrm{hocolim}_{\mathcal{C}} F^* Z \cong B(\mathcal{D} \downarrow F) \otimes_{\mathcal{D}} Z$.

Commutativity of hocolim. For a product diagram $W : \mathcal{C} \times \mathcal{D} \to \mathcal{U}$, $C \mapsto \mathrm{hocolim}_{\mathcal{D}} W(C, -)$ defines a $\mathcal{C}$-diagram, $\mathrm{hocolim}_{\mathcal{D}} W : \mathcal{C} \to \mathcal{U}$

$$\mathrm{hocolim}_{\mathcal{C}} \mathrm{hocolim}_{\mathcal{D}} W \cong \mathrm{hocolim}_{\mathcal{D}} \mathrm{hocolim}_{\mathcal{C}} W.$$

Commutativity of holim. With a product diagram as above

$$\mathrm{holim}_{\mathcal{C}} \mathrm{holim}_{\mathcal{D}} W \cong \mathrm{holim}_{\mathcal{D}} \mathrm{holim}_{\mathcal{C}} W.$$

One could also do this in $\mathcal{T}$. This is what is mainly used in Sections 1 and 2, it is then important that the spaces are nondegenerated.

For a product diagram $W : \mathcal{C} \times \mathcal{D} \to \mathcal{U}$, there is a natural map

$$\mathrm{hocolim}_{\mathcal{C}} \mathrm{holim}_{\mathcal{D}} W \longrightarrow \mathrm{holim}_{\mathcal{D}} \mathrm{hocolim}_{\mathcal{C}} W$$

This map is in general not an equivalence. According to [Good1] it is an equivalence when $\mathcal{C}$ is a filtered (directed) category and $\mathcal{D}$ is a finite category. A category is finite (in the strong sense) if the realization of its nerve is a compact space. This is the case when the nerve of the category is a finite simplicial set.

6 Appendix 2

We will here show that the map $(C\Omega B, \Omega B) \xrightarrow{\Phi} (PB, \Omega B)$ which appears in Proposition 4 is a relative homotopy equivalence. I will prove this in a slightly more general context, and divide the proof into three lemmas.

Lemma 34 *Let X be a contractible space. Given a map $f : S^n \to X$ there exists an extension of f to D^{n+1} for $n \geq 0$.*

Let $c : X \times I \to X$ be a homotopy $c : x_0 \simeq 1_X$, where x_0 is the point map to x_0. Regard D^{n+1} as $CS^n = S^n \times I/S^n \times 0$. Define the extension $CS^n \to X$ by the diagram

$$\begin{tikzcd} S^n \times I \arrow[r, "f \times 1_I"] \arrow[dr] & X \times I \arrow[r, "c"] & X. \\ & CS^n \arrow[ur] & \end{tikzcd}$$

□

This immediately extends to arbitrary disjoint unions: Any map $\amalg_\alpha S^n \xrightarrow{\amalg_\alpha f_\alpha} X$ extends to a map $\amalg_\alpha D^{n+1} \to X$ when X is contractible. Lemma 34 also hold when $n = -1$ by defining S^{-1} to be empty and D^0 to be a point.

We comment on the topology of a CW-complex. Let (X, A) be a relative CW-complex. $(X, A)^n$ is the n-skeleton consisting of A and all cells of dimension less then or equal to n. (X, A) has the coinduced topology from all finite subcomplexes. This is equivalent to the coinduced topology from $\{(X, A)^n\}$, where $(X, A)^n$ gets the coinduced topology from all the finite subcomplexes.

Lemma 35 *Let $\Phi : (X, A) \to (Y, B)$ be a map between relative CW-complexes. If X and Y are contractible spaces and $\Phi|A : A \to B$ is a homotopy equivalence, then Φ is a relative homotopy equivalence.*

Let $\psi : B \to A$ be a homotopy inverse of $\Phi|A$. We will first extend ψ to $\Psi : Y \to X$, and then extend homotopies.

Let α^n index all $(n+1)$-cells, and let $f_{\alpha^n} : S^n \to Y$ be the characteristic map of the cell α^n. ψ extends to a map $\Psi^0 : (Y, B)^0 = B \amalg \amalg_{\alpha^{-1}} y_{\alpha^{-1}} \to X$, just by giving a value at each point $y_{\alpha^{-1}}$. Assume ψ has been extended to $\Psi^n : (Y, B)^n \to X$, for some $n \geq 0$. The composition of the characteristic map $\amalg_{\alpha^n} S^n \to (Y, B)^n \xrightarrow{\Psi^n} X$ extends by Lemma 34 to a map $\amalg_{\alpha^n} D^{n+1} \to X$.

The map $\Psi^{n+1} : (Y, B)^{n+1} = (Y, B)^n \cup_{\amalg_{\alpha^n} f_{\alpha^n}} \amalg_{\alpha^n} D^{n+1} \to X$ is defined by the diagram

$$\begin{tikzcd} (Y, B)^n \amalg \amalg_{\alpha^n} D^{n+1} \arrow[d] \arrow[dr] & \\ (Y, B)^{n+1} = (Y, B)^n \cup_{\amalg_{\alpha^n} f_{\alpha^n}} \amalg_{\alpha^n} D^{n+1} \arrow[r] & X. \end{tikzcd}$$

By induction we get compatible extensions $\{\Psi^n\}$, hence a map

$$Y = \mathrm{colim}((Y,B)^0 \to (Y,B)^1 \to \cdots \to (Y,B)^n \to \cdots) \xrightarrow{\Psi} X.$$

Let $h : A \times I \to A$ be a homotopy $\psi \circ \Phi|A \simeq 1_A$. We will extend h to a homotopy $\Psi \circ \Phi \simeq 1_X$.

$$H^0 : (X,A)^0 \times I = A \amalg \amalg_{\alpha^{-1}} x_{\alpha^{-1}} \times I \to X$$

is defined by letting $H_t^0(x_{\alpha^{-1}})$ be a path from $\Psi \circ \Phi(x_{\alpha^{-1}})$ to $x_{\alpha^{-1}}$.

Assume that there exists a map $H^n : (X,A)^n \times I \cup X \times \{0,1\} \to X$ so that $H_0^n = \Psi \circ \Phi$ and $H_1^n = 1_X$.

The characteristic maps of all $(n+1)$-cells of X gives a map

$$\amalg_{\alpha^n} S^n \times I \cup D^{n+1} \times \{0,1\} \to (X,A)^n \times I \cup X \times \{0,1\} \xrightarrow{H^n} X.$$

Note that $D^{n+1} \times I$ is a $(n+2)$-cell, the boundary of this cell is $S^n \times I \cup D^{n+1} \times \{0,1\}$. It follows by Lemma 34, and as above, that the map extends to a map $H^{n+1} : (X,A)^{n+1} \times I \cup X \times \{0,1\} \to X$. So by induction we get a sequence $\{H^n\}$ of compatible extensions. Since $\mathrm{colim}((X,A)^{(n)} \times I) = \mathrm{colim}((X,A)^{(n)}) \times I$, we get the required homotopy. The same argument also shows that $\Phi|A \circ \psi \simeq 1_B$ extends to a homotopy $\Phi \circ \Psi \simeq 1_Y$. $\square$

Lemma 36 *Let $\Phi' : (X',A') \to (Y',B')$ be a map between spaces of the relative homotopy type of a CW-complex. If X' and Y' are contractible spaces and $\Phi'|A' : A' \to B'$ is a homotopy equivalence, then Φ' is a relative homotopy equivalence.*

By assumptions there are relative homotopy equivalences $(X',A') \simeq (X,A)$ and $(Y',B') \simeq (Y,B)$, where (X,A) and (Y,B) are relative CW-complexes. A relative homotopy equivalence is also a homotopy equivalence between the whole spaces, and between the subspaces. So both X and Y are contractible, and the map $\Phi : (X,A) \to (Y,B)$ defined by composing Φ' with the homotopy equivalences satisfies the requirements of Lemma 35. Hence Φ is a relative homotopy equivalence, and it follows that Φ' is a relative homotopy equivalence. $\square$

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