CONTINUITY OF π-PERFECTION FOR COMPACT LIE GROUPS

HALVARD FAUSK AND BOB OLIVER

Abstract. Let G be a compact Lie group, and let π be any prime or set of primes. We construct a "π-perfection map": a continuous function from the space of conjugacy classes of all closed subgroups of G to the space of conjugacy classes of π-perfect subgroups with finite index in their normalizer. We use this to show that the idempotent elements of the Burnside ring of G localized at π are in bijective correspondence with the open and closed subsets of the space of conjugacy classes of π-perfect subgroups of G with finite index in their normalizer.

1. π-perfection

Let π be a collection of primes, and let π' denote its complement. A discrete group H is *π-perfect* if it has no nontrivial solvable quotient π-groups. Any finite group H contains a unique maximal π-perfect subgroup, which we denote here H_π. Equivalently, H_π is the minimal normal subgroup of H such that H/H_π is a solvable π-group. It is easy to see that H_π is the terminal subgroup in the decreasing sequence of subgroups defined by setting $H_0 = H$, and letting H_n be the subgroup generated by the commutator $[H_{n-1}, H_{n-1}]$ and all π'-elements of H. All groups are $\varnothing$-perfect, while the $\{$all primes$\}$-perfect groups are exactly the perfect groups in the usual sense.

A compact Lie group H will be called π-perfect if the group $\pi_0(H) = H/H^\circ$ of its connected components is π-perfect. Hence the maximal π-perfect subgroup H'_π of an arbitrary compact Lie group H is the preimage in H of the maximal π-perfect subgroup of H/H°. When H is a closed subgroup in a compact Lie group G, there is a variant of this construction with better properties, where we replace H'_π by an associated subgroup H_π of G with finite index in its normalizer.

1991 *Mathematics Subject Classification.* Primary 55P91.
Key words and phrases. Burnside ring.
The second author was partially supported by UMR 7539 of the CNRS.

Let G be a compact Lie group. We give the space of closed subgroups of G the Hausdorff topology induced by any metric on G consistent with its topology. The topology is compact Hausdorff and independent of the metric.

Definition 1.1. *Let $\Psi(G)$ be the space of conjugacy classes of closed subgroups of G, regarded as a quotient space, with the quotient topology, of the space of all closed subgroups of G. Let $\Phi(G)$ be the subspace of $\Psi(G)$ consisting of conjugacy classes of subgroups of G with finite index in their normalizer.*

Both $\Psi(G)$ and $\Phi(G)$ are countable compact metric spaces, and hence totally disconnected. For any closed subgroup $H \leq G$, we let $(H) \in \Psi(G)$ denote its conjugacy class.

Given a subgroup $H \leq G$, there is a canonical way (up to conjugacy) to include H into a subgroup $K \leq G$ with finite index in its normalizer such that the quotient group K/H is a torus.

Definition 1.2. *Define*

$$\omega : \Psi(G) \longrightarrow \Phi(G)$$

as follows. For any $H \leq G$, let K/H be a maximal torus in $N_G(H)/H$, and set $\omega(H) = (K)$.

By [3, 5.7.5(ii)], the preimage in $N_G(H)$ of a maximal torus in $N_G(H)/H$ has finite index in its normalizer. So ω is well defined. The map ω is continuous (see the remarks after Lemma 2.2), and is a retraction of $\Psi(G)$ onto $\Phi(G)$.

Definition 1.3. *The* π-perfection *of a closed subgroup H in a compact Lie group G is $H_\pi \stackrel{\text{def}}{=} \omega(H'_\pi)$.*

We denote the π-perfection map by $P_\pi : \Psi(G) \longrightarrow \Phi(G)$. Note that H_π depends on the ambient group G, not only on H and π.

The map $\Psi(G) \longrightarrow \Psi(G)$ given by sending H to its maximal π-perfect subgroup H'_π is not continuous. The main result of this paper is the following theorem.

Theorem 1.4. *Let G be a compact Lie group. The π-perfection map*

$$P_\pi : \Psi(G) \longrightarrow \Phi(G)$$

is a continuous map.

Section 2 is devoted to a proof of this theorem. In Section 3, we give an application of the theorem to the Burnside ring $A(G)$ of a compact Lie group G. The G-equivariant cohomology theories are natural modules over $A(G)$. An idempotent element in $A(G)$ localized at π universally splits off a summand of all the G-equivariant cohomology theories localized to invert the set π of primes [7, XVII]. It is therefore important to describe the idempotent elements in $A(G)$ localized at π. Theorem 1.4 implies the following useful Lie group theoretic description of these elements. The idempotent elements in the Burnside ring $A(G)$, after localizing at π, are in bijective correspondence with open and closed subsets of the space of conjugacy classes of π-perfect subgroups of G with finite index in its normalizer.

2. Proof of the main theorem

In this section, we prove Theorem 1.4. We will need to refer to the following well known facts about compact Lie groups.

Proposition 2.1. *Let G be any compact Lie group.*

(1) (Montgomery & Zippin [8]) *For any sequence $\{H_i\}$ of subgroups of G which converges to some $H \leq G$, there are elements $g_i \in G$ such that $g_i \to e$ and $g_i H_i g_i^{-1} \leq H$ for all i sufficiently large.*

(2) *If there is a sequence $\{H_i\}$ of* finite *subgroups of G which converges to $H \leq G$, then H° is a torus.*

(3) *The group* $\mathrm{Out}(G)$ *of outer automorphisms of G is discrete.*

(4) *For any $H \trianglelefteq G$ such that G/H is a torus, $C_G(H)^\circ$ is a torus, and $G = C_G(H)^\circ \cdot H$.*

Proof. **(1)** This follows, for example, from [4, I.5.9]: for any neighborhood $U \subseteq G$ of e, there is a neighborhood V of H such that $K \subseteq V$ implies $gKg^{-1} \leq H$ for some $g \in U$.

(2) By (1), we can assume $H_i \leq H$ for all i. By Jordan's theorem [4, IV.6.4], there is some $j = j(H)$ such that every finite subgroup of H contains a normal abelian subgroup of index at most j. Choose abelian normal subgroups $A_i \trianglelefteq H_i$ of index less or equal to j. By the compactness of the space of subgroups of G, there is a subsequence $\{A_{i_j}\}$ which converges to A in the space of subgroups of H. Then A is an abelian normal subgroup of H, and $[H : A] \leq j$. So $H^\circ \leq A$ is a torus.

(3) Let $f_i \in \mathrm{Aut}(G)$ be a sequence of automorphisms converging to an automorphism f. Let $G_{f_i}, G_f \le G \times G$ denote the graphs of these maps. The sequence $\{G_{f_i}\}$ converges to G_f, so by (1), G_{f_i} is subconjugate (hence conjugate) to G_f for i sufficiently large. Hence f_i and f are equal in $\mathrm{Out}(G)$ for i large enough.

(4) Since $H \trianglelefteq G$, the group $G/(C_G(H){\cdot}H)$ is contained in $\mathrm{Out}(H)$, which is discrete by (3). Hence G and $C_G(H){\cdot}H$ have the same identity component. Since G/H is connected, this implies

$$G/H = (C_G(H){\cdot}H)/H \cong C_G(H)/Z(H) \cong C_G(H)^\circ/(Z(H) \cap C_G(H)^\circ).$$

So $G = C_G(H)^\circ{\cdot}H$, $C_G(H)^\circ/Z(H)^\circ$ is a connected finite covering group of a torus and hence a torus, and $C_G(H)^\circ$ is an extension of a torus by a torus and hence itself a torus. □

Lemma 2.2. *Let G be a compact Lie group, and let $H \le G$ be any closed subgroup. Then $\omega(H) = (TH)$ for any maximal torus T in $C_G(H)$.*

Proof. By definition, $\omega(H) = (K)$ for any $K \le N_G(H)$ such that K/H is a maximal torus of $N_G(H)/H$. Since K/H is a torus, Lemma 2.1(4) implies that $C_K(H)^\circ$ is a torus and $K = C_K(H)^\circ{\cdot}H$.

Let T be any maximal torus in $C_G(H)^\circ$. Then $C_K(H)^\circ$ is a torus in $C_G(H)^\circ$ and hence subconjugate to T, while TH/H is a torus in $N_G(H)/H$ and hence subconjugate to K/H. This shows that $K = C_K(H)^\circ{\cdot}H$ is conjugate to TH, and hence that $(TH) = (K) = \omega(H)$. □

The continuity of ω follows easily from Lemma 2.2. For any sequence $\{H_i\}$ of subgroups of G converging to some $H \le G$, we can assume $H_i \le H$ by (1), and hence $C_G(H_i) \ge C_G(H)$. The sequence of centralizers $\{C_G(H_i)\}$ converges to $C_G(H)$, since otherwise (after passing to a subsequence, using the compactness of G) there would be elements $g_i \in C_G(H_i)$ converging to $g \notin C_G(H)$, which is impossible. Proposition 2.1(1) then implies that $C_G(H_i) = C_G(H)$ for i sufficiently large. Hence for any maximal torus T of $C_G(H)$, $\omega(H_i) = (TH_i)$ (i large) by Lemma 2.2, and the sequence $\{(TH_i)\}$ converges to $(TH) = \omega(H)$.

Lemma 2.3. *Let G be a compact Lie group, and let $K \trianglelefteq H \le G$ be a pair of closed subgroups such that H/K is a torus. Then $\omega(H) = \omega(K)$.*

Proof. Set $S = C_H(K)^\circ$ for short; then S is a torus and $H = KS$ by Lemma 2.1(4). Let $T \le C_G(K)$ be any maximal torus which contains

S. Then T is also a maximal torus of $C_G(H) = C_G(KS)$, and $\omega(H) = (HT) = (KT) = \omega(K)$ by Lemma 2.2. □

We are now ready to prove the main theorem.

Proof of Theorem 1.4. Since every element of $\Psi(G)$ has a countable neighborhood basis, it suffices to show, for every sequence $\{H_i\}$ of closed subgroups of G which converges to a subgroup H, that there is a subsequence $\{H_{i_j}\}$ such that $\{P_\pi(H_{i_j})\}$ converges to $P_\pi(H)$. By Lemma 2.1(1) again, we can assume that $H_i \leq H$ for all i.

The space of closed subgroups of G is a compact metric space, so any sequence has an accumulation point. Hence after restricting to a subsequence, we can assume that $\{(H_i)'_\pi\}$ converges to some subgroup $\overline{H} \leq H$. Since $(H_i)'_\pi$ is normal in H_i for each i, it follows by taking limits that $\overline{H} \trianglelefteq H$.

Clearly, H_i surjects onto H/H° for i sufficiently large, and hence $(H_i)'_\pi$ surjects onto the π-perfect group H'_π/H°. So $\overline{H}$ surjects onto H'_π/H°, and in particular $H'_\pi/\overline{H}$ is connected.

Since $\overline{H}$ is normal in H, Lemma 2.1(1) tells us that $H_i^\circ \leq \overline{H}$ for i sufficiently large. In particular, the image K_i of H_i in $H/\overline{H}$ is a finite subgroup for i large, and the sequence $\{K_i\}$ converges to $H/\overline{H}$. By Lemma 2.1(2), this implies that $(H/\overline{H})^\circ$ is a torus, and hence (since $H'_\pi/\overline{H}$ is connected), that $H'_\pi/\overline{H}$ is a torus.

Thus $\omega(\overline{H}) = \omega(H'_\pi) = P_\pi(H)$ by Lemma 2.3. By the continuity of ω, the sequence $\{P_\pi(H_i)\} = \{\omega((H_i)'_\pi)\}$ converges to $\omega(\overline{H})$, and this finishes the proof of the theorem. □

3. Idempotents in the Burnside ring with π' inverted

For any ring R, we let $R_{(\pi)} = R \otimes_{\mathbb{Z}} \mathbb{Z}_{(\pi)}$ denote the localization of R at the set of primes π; i.e., R with the primes in the complement π' inverted. For example, $R_{(\{p\})} = R_{(p)}$: the localization of R at p.

The Burnside ring of a compact Lie group was defined by tom Dieck [3]. It generalizes the Burnside ring of a finite group; and (additively) can be regarded as the free group with basis the set of orbits G/K for all $(K) \in \Phi(G)$; i.e., all conjugacy classes of subgroups $K \leq G$ which have finite index in their normalizer. For each closed subgroup $H \leq G$, let $\phi_H\colon A(G) \longrightarrow \mathbb{Z}$ be the homomorphism $\phi_H(G/K) = \chi((G/K)^H)$. Let $C(\Phi(G), \mathbb{Z})$ be the ring of continuous integer valued functions on

$\Phi(G)$, and set

$$\phi = \big(\phi_H\big)_{H\in\Phi(G)} \colon A(G) \longrightarrow C(\Phi(G),\mathbb{Z}).$$

Then ϕ is injective, and identifies $A(G)$ as a subring of $C(\Phi(G),\mathbb{Z})$.

For each $H \leq G$, set $q(H,0) = \phi_H^{-1}(0)$ and (for any prime p) $q(H,p) = \phi_H^{-1}(p\mathbb{Z})$. If $H' \trianglelefteq H$ and H/H' is a torus, then clearly $\phi_{H'} = \phi_H$. Hence $q(H,0) = q(\omega(H),0)$ and $q(H,p) = q(\omega(H),p)$ for all H. The minimal prime ideals of $A(G)_{(\pi)}$ are precisely the ideals $q(H,0)$ for all conjugacy classes of subgroups H in $\Phi(G)$, and $q(H,0) = q(H',0)$ if and only if $(H) = (H')$ in $\Phi(G)$. The maximal ideals of $A(G)_{(\pi)}$ are the ideals $q(H,p)$ for all conjugacy classes $(H) \in \Phi(G)$ and all $p \in \pi$. Two maximal ideals $q(H,p)$ and $q(K,l)$ in $A(G)_{(\pi)}$ are equal if and only if $p = l$ and $(H_p) = (K_p)$ in $\Phi(G)$ (see [1, Prop. 8 & Theorem 4] or [7, XVII 3.3]). These are the only prime ideals. The closure of $q(H,0)$ in the Zariski topology consists of $q(H,0)$ and the $q(H,p)$ for all $p \in \pi$.

It is well known that the idempotent elements of a commutative unital ring R are in bijective correspondence with the open and closed subsets of the prime ideal spectrum $\operatorname{spec} R$. For any topological space X, let $\Pi_0(X)$ denote the space of components of X with the quotient topology from X. This is a totally disconnected Hausdorff space.

Definition 3.1. *Let $\Phi_\pi(G)$ denote the subspace of $\Phi(G)$ consisting of conjugacy classes of π-perfect subgroups of G with finite index in its normalizer.*

Note that $\Phi_\varnothing(G) = \Phi(G)$. Since the π-perfection map is continuous and $\Phi(G)$ is compact Hausdorff, we get the following.

Proposition 3.2. *The space $\Phi_\pi(G)$ of conjugacy classes of π-perfect subgroups of G with finite index in their normalizer is a closed subspace of $\Psi(G)$.*

We define a map $\beta : \Phi_\pi(G) \longrightarrow \Pi_0(\operatorname{spec} A(G)_{(\pi)})$ by sending the conjugacy class of H to the component of $q(H,0)$. As pointed out in [7, XVII.5.5] the continuity of the π-perfection map allows us to prove the following proposition.

Proposition 3.3. *The map*

$$\beta : \Phi_\pi(G) \longrightarrow \Pi_0(\operatorname{spec} A(G)_{(\pi)})$$

is a homeomorphism.

Proof. We already noted that for all $H \leq G$, $q(H,0)$ is in the same component as $q(H,p)$ for all primes $p \in \pi$. There is a sequence of normal extensions from H'_π to H with p-group quotients for various $p \in \pi$. From this, it follows that $q(H, 0)$ is in the same component as $q(H'_\pi, 0)$. Since H_π/H'_π is a torus, $q(H_\pi, 0) = q(H'_\pi, 0)$.

The map $\alpha' : \operatorname{spec} A(G)_{(\pi)} \longrightarrow \Phi_\pi(G)$, defined by sending $q(H, p)$ and $q(H, 0)$ to H_π, is well defined. The composite map

$$\Phi(G) \times \operatorname{spec} \mathbb{Z}_{(\pi)} \xrightarrow{\quad q \quad} \operatorname{spec} A(G)_{(\pi)} \xrightarrow{\quad \alpha' \quad} \Phi_\pi(G)$$

is continuous since it is equal to the composite

$$\Phi(G) \times \operatorname{spec} \mathbb{Z}_{(\pi)} \xrightarrow{\quad \mathrm{pr}_1 \quad} \Phi(G) \xrightarrow{\quad P_\pi \quad} \Phi_\pi(G)$$

of the projection and π-perfection maps. The map

$$q : \Phi(G) \times \operatorname{spec} \mathbb{Z}_{(\pi)} \longrightarrow \operatorname{spec} A(G)_{(\pi)}$$

is an identification [6, V.5.7]. So α' is continuous. Since $\Phi_\pi(G)$ is totally disconnected, we get that α' factors through the space of components of $\operatorname{spec} A(G)_{(\pi)}$. This gives a continuous map

$$\alpha : \Pi_0(\operatorname{spec} A(G)_{(\pi)}) \longrightarrow \Phi_\pi(G)$$

that sends the component containing $q(H, 0)$ to H_π. The maps α and β are inverses of each other. Also, β is continuous since q is continuous. Hence α and β are mutually inverse homeomorphisms. □

In the case $\pi = \varnothing$, this result was proved by tom Dieck [2]. Proposition 3.3 gives the following description of the idempotent element of $A(G)$ localized at a set of primes.

Theorem 3.4. *There is a bijection between open and closed subsets of $\Phi_\pi(G)$ and idempotent elements of $A(G)_{(\pi)}$. Let e_U denote the idempotent element of $A(G)_{(\pi)}$ corresponding to an open and closed subset U of $\Phi_\pi(G)$. The image of e_U in $C(\Phi(G), \mathbb{Z}_{(\pi)})$ is described by $\phi_H(e_U) = 1$ if $H_\pi \in U$, and $\phi_H(e_U) = 0$ if $H_\pi \notin U$.* □

Note that Lemma 2.1(1) implies that the conjugacy class of any abelian subgroup of G with finite index in its normalizer is an open and closed point in $\Phi(G)$. This observation, together with Theorem 3.4, is used in [5].

References

[1] T. tom Dieck, The Burnside ring of a compact Lie group I, Math. Ann. 215 (1975), 235–250.
[2] T. tom Dieck, Idempotent elements in the Burnside ring, J. Pure Appl. Algebra 10 (1977/78), no. 3, 239–247.

[3] T. tom Dieck, Transformation groups and representation theory, Lecture Notes in Math. Vol 766. Springer Verlag. 1979.
[4] T. tom Dieck, Transformation groups, Walter de Gruyter. 1987.
[5] H. Fausk, Generalized Artin and Brauer induction for compact Lie groups, preprint 2001.
[6] L. G. Lewis, J. P. May, and M. Steinberger (with contributions by J. E. McClure), Equivariant stable homotopy theory, Lecture Notes in Math. Vol 1213. Springer Verlag. 1986.
[7] J. P. May, et. al., Equivariant homotopy and cohomology theories. CBMS regional conference series no. 91, Amer. Math. Soc. 1996.
[8] D. Montgomery & L. Zippin, A theorem on Lie groups, Bull. Amer. Math. Soc. 48 (1942), 448–452.

Department of Mathematics, University of Oslo, Norway

E-mail address: fausk@math.uio.no

LAGA, Institut Galilée, Av. J-B Clément, 93430 Villetaneuse, France

E-mail address: bob@math.univ-paris13.fr