THE UNIVERSITY OF CHICAGO

THE NON-POSITIVE EQUIVARIANT STEMS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS
JUNE 2001

ABSTRACT

Part I: The G-equivariant analogues of the stable homotopy groups of spheres are the equivariant homotopy groups of stable homotopy representations. Let G be a compact Lie group and $A(G)$ the Burnside ring of G. Let Z be a stable homotopy representation with non-negative dimension function. We prove, with one extra hypothesis on Z, that $\pi_0^G(Z)$, as an $A(G)$-module, is isomorphic to a quotient of $A(G)$ tensored with an invertible $A(G)$-module. This is the equivariant analogue of the non-positive stems.

Part II: We investigate the Picard group $\mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$ of isomorphism classes of invertible objects in the derived category of $\mathcal{O}$-modules for a commutative unital ringed Grothendieck topos $(\mathcal{E}, \mathcal{O})$ with enough points. Let $C(\mathrm{pt}(\mathcal{E}))$ denote the additive group of continuous functions from the space of isomorphism classes of points of $\mathcal{E}$ to the integers. When the ring $\mathcal{O}_p$ has connected prime ideal spectrum for all points p of $\mathcal{E}$ we show that $\mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$ is naturally isomorphic to the Cartesian product of $C(\mathrm{pt}(\mathcal{E}))$ with the Picard group of $\mathcal{O}$-modules. Also, for a commutative unital ring R, the group $\mathrm{Pic}(\mathcal{D}_R)$ is isomorphic to the Cartesian product of $\mathrm{Pic}(R)$ and the additive group of continuous functions from $\operatorname{spec} R$ to the integers.

ACKNOWLEDGMENTS

I would like to thank the Department of Mathematics at the University of Chicago for admitting me to their graduate program and for funding me throughout the years. I am thankful to faculty, students and the administrative personnel for creating a friendly and stimulating environment in which to live and work.

I would like to express my deepest gratitude to my advisor J. Peter May for everything he taught me. Most of all I am thankful to him for his understanding and support during my second year.

There are many people I have learned from and been inspired by. In particular I would like to thank J. Alperin, M. Emerton, C. French, G. Glauberman, J. P. C. Greenlees, G. Lewis, M. Mandell, J. Morava, R. Narasimhan, M. Nori, and my master thesis advisor J. Rognes.

I would also like to thank my parents, my sister, and Jeanne M. Haffner.

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INTRODUCTION

Part I

Let G be a compact Lie group. A homotopy representation X is a based homotopy retract of a finite G-CW complex such that the fixed-point space X^H is nonequivariantly homotopy equivalent to a sphere for every subgroup H of G. In the G-equivariant stable homotopy category, the invertible G-spectra are exactly the stable homotopy representations $\Sigma^{-V}\Sigma_G^{\infty}X$, where V is a real G-representation and X is a homotopy representation. The G-equivariant analogues of the stable homotopy groups of spheres are the G-equivariant homotopy groups of stable homotopy representations. A fundamental theorem due to G. Segal says that for finite groups G the ring $\pi_0^G(S_G^0)$ of homotopy classes of maps from the stable equivariant sphere spectrum to itself is isomorphic to the Burnside ring $A(G)$. T. tom Dieck gave a geometric definition of the Burnside ring of any compact Lie group G and proved that it is isomorphic to $\pi_0^G(S_G^0)$. All G-equivariant stable homotopy groups are naturally modules over the Burnside ring $A(G)$.

The dimension function of a stable homotopy representation $Z = \Sigma^{-V}\Sigma_G^{\infty}X$ is defined to be $\dim(Z)(H) = n(H) - \dim_{\mathbb{R}} V^H$ where $X^H \simeq S^{n(H)}$. T. tom Dieck and T. Petrie have proven that when Z is a stable homotopy representation with dimension function identically zero, then the 0-th G-equivariant stable homotopy group $\pi_0^G(Z)$ is an invertible $A(G)$-module. Furthermore, the map sending Z to $\pi_0^G(Z)$ induces an isomorphism between the group of isomorphism classes of stable homotopy representations with dimension function identically zero and the group of isomorphism classes of invertible $A(G)$-modules.

Let Z be a stable homotopy representation with nonnegative dimension function. Let $I(Z)$ be the ideal of $A(G)$ consisting of all elements that act trivially on $\pi_0^G(Z)$.

In particular $\pi_0^G(Z)$ is an $A(G)/I(Z)$-module. Our main theorem gives the following description of the equivariant non-positive stems as modules over the Burnside ring for any compact Lie group G. If $\dim(Z) \geq 0$, and the set of subgroups of G where the dimension of Z is zero forms a cofamily, then there is an isomorphism of $A(G)$-modules

$$\pi_0^G(Z) \cong A(G)/I(Z) \otimes_{A(G)} P$$

where P is an invertible $A(G)$-module.

We prove the theorem by showing that $\pi_0^G(Z)$ is an invertible $A(G)/I(Z)$-module and that the invertible $A(G)/I(Z)$-module $\pi_0^G(Z)$ comes from an invertible $A(G)$-module. More precisely, for every maximal ideal $\mathfrak{m}$ of $A(G)/I(Z)$, we use obstruction theory to construct a map $f : S_G^0 \to Z$ that we prove induces an isomorphism

$$f_* : A(G)_{\mathfrak{m}} \to \pi_0^G(Z)_{\mathfrak{m}}$$

localized at $\mathfrak{m}$. We then show that $\pi_0^G(Z)$ is a finitely generated $A(G)$-module. This implies that $\pi_0^G(Z)$ is an invertible $A(G)/I(Z)$-module. Finally, we prove that for the ideals $I(Z)$ of $A(G)$ the homomorphism of Picard groups

$$\mathrm{Pic}(A(G)) \to \mathrm{Pic}(A(G)/I(Z))$$

given by tensoring with the quotient ring $A(G)/I(Z)$ is surjective. In particular, the invertible $A(G)/I(Z)$-module $\pi_0^G(Z)$ is isomorphic to $A(G)/I(Z) \otimes_{A(G)} P$ for some invertible $A(G)$-module P.

There is a similar description for any stable homotopy representation rationally. When G is a finite group and Z is a stable homotopy representation it is easily proved that

$$\pi_0^G(Z) \otimes_{\mathbb{Z}} \mathbb{Q} \cong A(G)/I(Z) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Here $I(Z)$ is the ideal of $A(G)$ consisting of elements that act trivially on $\pi_0^G(Z) \otimes \mathbb{Q}$. This is true for any compact Lie group provided that Z has non-negative dimension function.

Part II

The Picard group of the derived category for a commutative unital ring has been calculated in some cases. We give a complete description of the Picard group of the unbounded derived category for any commutative unital ring.

More generally, we calculate the Picard group of the derived category for any commutative unital ringed Grothendieck topos $(\mathcal{E}, \mathcal{O})$ with enough points. That $\mathcal{E}$ is a Grothendieck topos means that there is a site $\mathcal{C}$ such that $\mathcal{E}$ is equivalent to the category of sheaves of sets on the site $\mathcal{C}$. We assume that $\mathcal{C}$ is small, and let $\mathrm{pt}(\mathcal{E})$ denote the set of isomorphism classes of points of $\mathcal{E}$. We let $\mathcal{M}$ denote the category of left $\mathcal{O}$-modules in $\mathcal{E}$. The derived category $\mathcal{D}_{\mathcal{M}}$ is obtained from the category of cochain complexes of left $\mathcal{O}$-modules by formally inverting cochain maps that induce an isomorphism on cohomology. It is a symmetric tensor category under the derived tensor product.

Let $(\mathcal{T}, \otimes, I)$ be a symmetric tensor category. An object X in $\mathcal{T}$ is said to be invertible if there exists an object Y such that $X \otimes Y$ is isomorphic to the unit object I. The Picard group $\mathrm{Pic}(\mathcal{T})$ of $\mathcal{T}$ is defined to be the group of isomorphism classes of invertible objects in $\mathcal{T}$, when this is a set. The multiplication in $\mathrm{Pic}(\mathcal{T})$ is the tensor product, and the class of the unit object is the unit element.

Let R be a commutative unital ring, and let $\mathcal{D}_R$ denote the derived category of complexes of left R-modules. For a topological space T let $C(T)$ denote the additive group of continuous functions from T to the integers endowed with the discrete topology. We prove that there is a natural split short exact sequence

$$0 \to \mathrm{Pic}(R) \to \mathrm{Pic}(\mathcal{D}_R) \xrightarrow{\Psi} C(\operatorname{spec} R) \to 0.$$

Let $M^\bullet$ be an invertible complex of R-modules. The function $\Psi(M^\bullet)$ sends a prime ideal $\mathfrak{p}$ to the unique integer n such that $\mathrm{H}^n(M^\bullet)_{\mathfrak{p}} \neq 0$.

More generally, for $(\mathcal{E}, \mathcal{O})$ as above, if the ring $\mathcal{O}_p$ has a connected prime ideal spectrum for all points p of $\mathcal{E}$, then there is a natural split short exact sequence

$$0 \to \mathrm{Pic}(\mathcal{M}) \to \mathrm{Pic}(\mathcal{D}_{\mathcal{M}}) \xrightarrow{\Psi} C(\mathrm{pt}(\mathcal{E})) \to 0.$$

In general, let $C(\mathcal{O})$ be the sheaf associated to the presheaf which maps an object X to $C(\operatorname{spec}\mathcal{O}(X))$. We prove that there is a natural split short exact sequence

$$0 \to \operatorname{Pic}(\mathcal{M}) \to \operatorname{Pic}(\mathcal{D}_{\mathcal{M}}) \stackrel{\Psi}{\to} \Gamma(C(\mathcal{O})) \to 0$$

where Γ denotes the global sections functor.

Part I

The Non-Positive Equivariant Stems

CHAPTER 2
THE NON-POSITIVE EQUIVARIANT STEMS

2.1 Introduction

Let G be a compact Lie group. A homotopy representation X is a based homotopy retract of a finite G-CW complex such that the fixed-point space X^H is nonequivariantly homotopy equivalent to a sphere for every closed subgroup H of G. In the G-equivariant stable homotopy category the invertible G-spectra are exactly the stable homotopy representations $\Sigma^{-V}\Sigma^\infty_G X$, where V is a real G-representation and X is a homotopy representation [8]. We use the letter Z to denote stable homotopy representations. The G-equivariant analogues of the stable homotopy groups of spheres are the G-equivariant stable homotopy groups $\pi_0^G(Z)$ of stable homotopy representations. The groups $\pi_0^G(Z)$ are naturally modules over the Burnside ring $A(G) \cong \pi_0^G(S^0_G)$. We describe $\pi_0^G(Z)$ as a module over the Burnside ring for certain homotopy representations. For a survey on the Burnside ring for compact Lie groups and its prime ideals, see [16, Chap. 17].

Let ΦG denote the topological space of conjugacy classes of closed subgroups H of G with finite Weyl group $WH = N_G H/H$. It is a countable metric space, hence has a basis for the topology consisting of open-closed sets. T. tom Dieck proved that for any compact Lie group G, the orders $|WH|$ are uniformly bounded for all $(H) \in \Phi G$ [4]. Let the order of a compact Lie group G be the least common multiple of the order of all the Weyl groups WH for $(H) \in \Phi G$. We denote the order of G by $|G|$.

We associate to any stable homotopy representation $Z = \Sigma^{-V}\Sigma^\infty_G X$ a subspace of ΦG and an ideal of $A(G)$.

Definition 2.1.1. Let $\mathcal{E}(Z)$ be the following subspace of ΦG:

$$\mathcal{E}(Z) = \{(K) \in \Phi G \mid H^0(Z^K) \cong \mathbb{Z} \text{ with trivial } WK\text{-action}\}.$$

In other words, $(K) \in \mathcal{E}(Z)$ if and only if $S^{V^K} \simeq S^{n(K)} \simeq X^K$ for some integer $n(K)$, and $H^{n(K)}(S^{V^K}) \cong H^{n(K)}(X^K)$ as WK-modules. When $|G|$ is odd $\mathcal{E}(Z) = \{(K) \in \Phi G \mid S^{V^K} \simeq X^K\}$ since the finite groups $|WK|$ are then of odd order, and hence can only act trivially on $\mathbb{Z}$.

A cofamily of subgroups of G is a collection of closed subgroups of G that is closed under conjugation and passing to larger subgroups. The class of subgroups of G with finite Weyl group is a cofamily of subgroups of G [2, II.5.7]. Hence we say that $\mathcal{E}(Z)$ is a cofamily if whenever $(K) < (H)$ and $(K) \in \mathcal{E}(Z)$, then $(H) \in \mathcal{E}(Z)$. It turns out that if

$$\mathcal{E}'(Z) = \{\, (K) \in \Phi G \mid X^K \simeq (S^V)^K \,\}$$

is a cofamily, then we have that $\mathcal{E}(Z) = \mathcal{E}'(Z)$.

The Burnside ring of any compact Lie group has Krull dimension one. The minimal prime ideals of $A(G)$ are the ideals $q(H,0)$, for any $(H) \in \Phi G$, consisting of all stable homotopy classes of maps $\alpha : S^0_G \to S^0_G$ such that $\deg(\alpha^H) = 0$. The maximal prime ideals of $A(G)$ are the ideals $q(H,p)$, for any $(H) \in \Phi G$ and prime number p, consisting of all stable homotopy classes of maps $\alpha : S^0_G \to S^0_G$ such that $\deg(\alpha^H) \equiv 0 \bmod p$. Here, and later, α^H denotes the geometrical fixed point functor applied to the map α [14, II.9]. The geometrical fixed point map of a stable map from a finite G-CW complex to a G-space is obtained by taking the pointwise fixed point map of some unstable representative of the map.

Definition 2.1.2. Let $\mathcal{E}$ be any subspace of ΦG. Define the $A(G)$-ideal $I(\mathcal{E})$ to be

$$I(\mathcal{E}) = \bigcap_{(H)\in\mathcal{E}} q(H,0).$$

The ideal $I(\mathcal{E})$ consists of all stable maps $\alpha : S^0_G \to S^0_G$ such that the degree of α^H is zero whenever $(H) \in \mathcal{E}$. We denote for brevity $I(\mathcal{E}(Z))$ by $I(Z)$. The dimension function $\dim(Z) : \Phi G \to \mathbb{Z}$ is defined by $\dim(Z)(H) = n(H) - \dim V^H$ where $X^H \cong S^{n(H)}$. When the dimension function $\dim(Z) \geq 0$ we have that $\pi^G_0(Z)$ is an $A(G)/I(Z)$-module. This follows since $I(Z)\pi^G_0(Z)$ consists of stable G-maps with null-homotopic fixed-point maps for each subgroup of G. By obstruction theory such G-maps are null-homotopic (Theorem 2.2.4).

Nonequivariantly $\pi_0(S^n) = 0$ when $n > 0$ and $\pi_0(S^0) = \mathbb{Z}$. We prove the following, partial, G-equivariant analogue.

Theorem 2.1.3. *Let G be a compact Lie group, and let Z be a stable homotopy representation with dimension function $dim(Z) \geq 0$ and $\mathcal{E}(Z)$ a cofamily. Then there is an isomorphism of $A(G)$-modules*

$$\pi_0^G(Z) \cong A(G)/I(Z) \otimes_{A(G)} P$$

where P is an invertible $A(G)$-module.

Since $H^*(V^K; \mathbb{Z})$ and $H^*(X^K; \mathbb{Z})$ are isomorphic as WK-modules whenever $(K) \in \mathcal{E}(Z)$, and since we can assume that V is a complex representation containing a copy of the trivial representation, we get the following result as in [5, 10.2.5].

Proposition 2.1.4. *Let Z be a stable homotopy representation with dimension function* $\dim(Z) \geq 0$. *Let $(H) \in \mathcal{E}(Z)$ with $|WH|$ relatively prime to p. Then there is a G-map $f : S_G^0 \to Z$ such that $deg(f^H)$ is relatively prime to p.*

The main part of this paper, section 2.2, is devoted to prove that for any f as in the Proposition above, the induced map localized at $q(H, p)$

$$f_* : (A(G)/I(Z))_{q(H,p)} \to \pi_0^G(Z)_{q(H,p)}.$$

is an isomorphism. When $\mathcal{E}(Z)$ is a cofamily, we show that any maximal ideal in $A(G)/I(Z)$ is of the form $q(H, p)$ where $(H) \in \mathcal{E}(Z)$ and $|WH|$ is relative prime to p. Hence we get that $\pi_0^G(Z)$ is locally isomorphic to $A(G)/I(Z)$. We also prove that $\pi_0^G(Z)$ is a finitely generated $A(G)$-module. Together this gives the following result.

Theorem 2.1.5. *Let G be a compact Lie group. Let Z be a stable homotopy representation with dimension function $dim(Z) \geq 0$ and $\mathcal{E}(Z)$ a cofamily. Then $\pi_0^G(Z)$ is an invertible $A(G)/I(Z)$-module.*

In section 2.3 we prove the following result.

Proposition 2.1.6. *Let Z be a stable homotopy representation with $\mathcal{E}(Z)$ closed in ΦG. Then any invertible $A(G)/I(Z)$-module is isomorphic to $A(G)/I(Z) \otimes_{A(G)} P$ for some invertible $A(G)$-module P.*

Hence when $\mathcal{E}(Z)$ is a cofamily the invertible $A(G)/I(Z)$-module $\pi_0^G(Z)$ is isomorphic to

$$A(G)/I(Z) \otimes_{A(G)} P$$

for some invertible $A(G)$-module P. This is Theorem 2.1.3.

It is natural to make the following conjecture.

Conjecture 2.1.7. *For any stable homotopy representation Z with non-negative dimension function $dim(Z) \geq 0$, there is an isomorphism of $A(G)$-modules*

$$\pi_0^G(Z) \cong A(G)/I(Z) \otimes_{A(G)} P$$

where P is an invertible $A(G)$-module.

In section 2.4 we give some evidence for the conjecture. We prove that the conjecture is true if we invert the order of the group, provided that $\mathcal{E}(Z)$ is a closed subspace of ΦG. We also quote a theorem from [12, 14] describing $\pi_0^G(Z) \otimes_{\mathbb{Z}} \mathbb{Z}[|G|^{-1}]$ when G is a finite group and Z is any stable homotopy representation.

The subspace $\mathcal{E}(Z)$ of ΦG is closed when G is a finite group (since ΦG is discrete). It is closed for all compact Lie groups of odd order (since the dimension function is locally constant). If $\mathcal{E}(Z)$ is a cofamily then $\mathcal{E}(Z)$ is also closed. One might expect $\mathcal{E}(Z)$ to be closed for any compact Lie group and any stable homotopy representation. In general $\mathcal{E}(Z)$ is always open.

In the last section, included for completeness, we generalize, and give an alternative proof of Proposition 2.1.6. When G is a finite group we show that for any ideal I of $A(G)$ the map

$$\mathrm{Pic}(A(G)) \to \mathrm{Pic}(A(G)/I)$$

given by tensoring with $A(G)/I$ is surjective.

2.2 Non-positive Equivariant Stems

This section is concerned with the proof of Theorem 2.1.5. Let $Z = \Sigma^{-V}\Sigma_G^\infty X$ be a stable homotopy representation with $\dim Z \geq 0$. We can assume without loss of generality that V is a complex representation containing a copy of the trivial representation. We let V and X vary as needed (suspend up with a representation), in order to realize any given stable map $S_G^0 \to Z$ as the stabilization of an unstable map $S^V \to X$.

For every $(H) \in \mathcal{E}(Z)$ choose an orientation of the fixed point spaces of S^V and X. For any G-map $f : S^V \to X$, let $\deg(f) : \mathcal{E}(Z) \to \mathbb{Z}$ be the function that sends (H) to the degree of the map $f^H : S^{V^H} \to X^H$. Since $\dim(Z) \geq 0$ the fixed point map $f^H : S^{V^H} \to X^H$ is null homotopic if $(H) \notin \mathcal{E}(Z)$. We only use the absolute value of the degree function. Hence the choice of orientations does not matter. From the proof of [5, 5.6.4], we get that for all $f : S^V \to X$ the absolute value of the degree function

$$|\deg(f)| : \mathcal{E}(Z) \to \mathbb{N}$$

is locally constant.

For two based G-spaces X and X' let $[X, X']$ denote the pointed set of homotopy classes of G-maps from X to X'. We only consider spaces such that this is an abelian group. For any two based G-spaces X and X' let $\{X, X'\}$ denote the abelian group of stable homotopy classes of G-maps from X to X'. When X is a retract of a finite G-CW complex

$$\{X, X'\} \cong \operatorname{colim}_U [X \wedge S^U, X' \wedge S^U]$$

where the direct limit is over all complex G-representations U. In particular any stable map between homotopy representations can be realized by a suitable unstable map.

Lemma 2.2.1. *The subspace $\mathcal{E}(Z)$ is an open subset of ΦG.*

Proof. Let $(H) \in \mathcal{E}(Z)$. Then by Proposition 2.1.4 there is a map $f : S^V \to X$ such that $\deg(f^H) \neq 0$. Both S^V and X have finite orbit types [13]. Hence there is an open neighborhood U of (H) in $\Phi(G)$ such that for all $(K) \in U$ the map f^K equals the map f^H. Hence $\deg(f^K) \neq 0$ for all $(K) \in U$, and $U \subset \mathcal{E}(Z)$. □

Lemma 2.2.2. *If $\mathcal{E}(Z)$ is a cofamily, then $\mathcal{E}(Z)$ is a closed subset.*

Proof. A theorem of Montgomery and Zippin says that for any closed subgroup H of G there is an open neighborhood of H, in the space of all closed subgroups, consisting entirely of subgroups of G subconjugate to H. Hence the complement of any cofamily is open. □

Let $\mathcal{E}$ be a subset of ΦG. We now describe the prime ideals of $A(G)/I(\mathcal{E})$.

Lemma 2.2.3. *The prime ideals of $A(G)$ containing $I(\mathcal{E}) = \bigcap_{(H)\in\mathcal{E}} q(H,0)$ are exactly the prime ideals $q(H,0)$ and $q(H,p)$ of $A(G)$ where p is any prime number and $(H) \in \bar{\mathcal{E}}$, the closure of $\mathcal{E}$ in ΦG.*

Proof. Let $\mathcal{E}' = \{(K) \in \Phi G \mid q(K,0) \supset I(\mathcal{E})\}$. Clearly $\mathcal{E}' \supset \mathcal{E}$ and $I(\mathcal{E}') = I(\mathcal{E})$. If $(K) \notin \mathcal{E}'$ then there is an $\alpha \in A(G)$ such that $\deg \alpha^K \neq 0$ and $\deg \alpha^H = 0$ for all $(H) \in \mathcal{E}$. Since $|\deg|$ is a locally constant function there is an open neighborhood U around (K) in ΦG where $|\deg \alpha|$ is nonzero, so U does not meet $\mathcal{E}'$. Hence $\mathcal{E}'$ is a closed subset of ΦG.

If $(K) \notin \bar{\mathcal{E}}$, then there is an open-closed neighborhood U of (K) that does not meet $\bar{\mathcal{E}}$. Since $|G|C(G) \subset A(G)$ (see [16, chap. 17]) we obtain that the characteristic function in $C(G)$, defined to be $|G|$ on U and 0 outside of U, is in $A(G)$. Hence $(K) \notin \mathcal{E}'$. We conclude that $\mathcal{E}' = \bar{\mathcal{E}}$. □

The lemma gives that the prime ideals of $A(G)/I(\mathcal{E})$ are exactly the quotients in $A(G)/I(\mathcal{E})$ of the prime ideals $q(H,0)$ and $q(H,p)$ of $A(G)$, where p is any prime number and $(H) \in \bar{\mathcal{E}}$. We use the same notation for a prime ideal of $A(G)/I(Z)$ as for the corresponding prime ideal of $A(G)$.

For any complex G-representation V containing a copy of the trivial representation the assumption of the equivariant Hopf Theorem [5, 8.4.1] is satisfied for S^V, and we get the following result. Let $\mathrm{iso}(V)$ denote the isotropy classes of S^V.

Theorem 2.2.4. *The degree function*

$$deg\ : [S^V, X] \to \prod_{(H)\in iso(V)\cap\mathcal{E}(Z)} \mathbb{Z}$$

is injective. Let $H \in iso(V) \cap \mathcal{E}(Z)$ *and* $f \in [S^V, X]$. *Then for any* $g \in [S^V, X]$ *with* $deg\ (f^K) = deg\ (g^K)$ *for all* $(K) \in iso(V) \cap \mathcal{E}(Z)$ *with* $(K) > (H)$, *we have that*

$$deg\ (f^H) \equiv deg\ (g^H)\ mod\ |WH|.$$

Moreover given f *and any integer* n, *there is a* $g \in [S^V, X]$ *such that* $deg\ (f^K) = deg\ (g^K)$ *for all* $(K) \in iso(V) \cap \mathcal{E}(Z)$ *with* $(K) > (H)$ *and*

$$deg\ (f^H) - deg\ (g^H) = n|WH|.$$

In particular the degree map

$$\deg\ : \{S^V, X\} \to \prod_{\mathcal{E}(Z)} \mathbb{Z}$$

is an injective homomorphism.

We now digress to calculate the homotopy groups of stable homotopy representations of the form $\Sigma_G^\infty S^W$ for any real representation W.

Lemma 2.2.5. *Assume there is a map* $f : S^V \to X$ *such that* $deg(f^H) = \pm 1$ *for all* $(H) \in \mathcal{E}(Z)$. *Then*

$$f_* : \{S^V, S^V\}/I(\mathcal{E}) \to \{S^V, X\}$$

is an isomorphism of $A(G)$*-modules.*

Proof. We first prove that we get an isomorphism unstably. The statement in the lemma follows by stabilizing, replacing S^V, X by $S^{V+U}, X \wedge S^U$, and then taking the direct limit over all complex G-representations U.

Let I_V be the ideal of the ring $[S^V, S^V]$ consisting of maps α with $\deg(\alpha^K) = 0$ for all $(K) \in \mathcal{E}(Z)$. Let the map $f : S^V \to X$ induce the horizontal homomorphisms in

$$\begin{array}{ccc} [S^V, S^V]/I_V & \xrightarrow{f_*} & [S^V, X] \\ \Big\downarrow{\scriptstyle \deg} & & \Big\downarrow{\scriptstyle \deg} \\ \prod_{(H)\in \mathrm{iso}(V)\cap \mathcal{E}(Z)} \mathbb{Z} & \xrightarrow{\deg(f)} & \prod_{(H)\in \mathrm{iso}(V)\cap \mathcal{E}(Z)} \mathbb{Z}. \end{array}$$

The degree functions are injective, hence $[S^V, S^V]/I_V \to [S^V, X]$ is injective. By the congruence relations, from the above theorem, and induction on the orbit type of S^V, we get that $[S^V, S^V]/I_V \to [S^V, X]$ is surjective. Hence we conclude that the map f_* is an isomorphism. □

For a real representation W, the inclusion $S^0 \to S^W$ satisfies the requirement in lemma 2.2.5.

Corollary 2.2.6. *Let W be a real representation. Then there is an isomorphism*

$$\pi_0^G(\Sigma_G^\infty S^W) \cong A(G)/I(\Sigma_G^\infty S^W).$$

Let $\Phi(H,p)$ be the subspace of ΦG consisting of all (J) such that $q(H,p) = q(J,p)$. For every (H) and prime p there is a unique element $(K) \in \Phi(H,p)$ such that $|WK|$ is relatively prime to p [5, 5.7.2].

Proposition 2.2.7. *Let Z be a stable homotopy representation such that $\mathcal{E}(Z)$ is a cofamily and $dim(Z) \geq 0$. Let $(H) \in \mathcal{E}(Z)$ be such that $|WH|$ is prime to p. Then for any stable map $f : S_G^0 \to Z$ such that the degree of f^H is prime to p, the degree of f^K is also prime to p for all $(K) \in \mathcal{E}(Z) \cap \Phi(H,p)$.*

Proof. Step 1: Assume that (K) and (L) are in $\mathcal{E}(Z) \cap \Phi(H,p)$ and that $K \lhd L$ with quotient group L/K a p-group. We then prove that if the degree of f^L is prime to p, then the degree of f^K is also prime to p. We denote L/K by P.

The geometric fixed point functor and the restriction to subgroup functor both preserve smash products of spectra and sphere spectra [14, II.9]. The stable homotopy representation Z is an invertible G-spectrum [8, 0.5]. This means that there is a G-spectrum Y such that $Z \wedge Y \simeq S_G^0$. The K-geometric fixed point functor applied to this equivalence gives that $Z^K \wedge Y^K \simeq S^0_{WK}$. Restriction to the subgroup $P = L/K < WK$ gives that Z^K is an invertible P-spectrum.

Whenever $K < H$ and $|WK|$ is finite then $|WH|$ is also finite [2, II.5.7]. Hence, since $\mathcal{E}(Z)$ is a cofamily, we have that any closed subgroup J between K and L is also in $\Phi(H,p) \cap \mathcal{E}(Z)$ [16, 17.3.4]. So the dimension of Z^J is zero for all subgroups J between K and L. All invertible spectra are stable homotopy representations, so the P-spectrum Z^K is a stable homotopy representation with identically zero dimension function [8]. As in [8, 3.4] we have that there is a map of P-spectra $h : Z^K \to S^0_P$ such that the degree of $(h \circ f^K)^P$ is prime to p (note that P as a subgroup of itself has trivial Weyl group), hence $h \circ f^K \in q(P,p)$. For any subgroup Q of the p-group P we have that $q(Q,p) = q(P,p)$ as ideals in $A(P)$ [5, 5.7.9]. In particular, when $Q = \{0\}$ we get that the degree of f^K is also prime to p.

Step 2: We now consider the general case. We follow tom Dieck [5, p. 114]. Let H be a subgroup of G with WH prime to P. For every $(K) \in \Phi G$ with $q(H,p) = q(K,p)$ there is a finite sequence

$$K = K_0 < K_1 < K_2 < K_3 < \cdots < K_n$$

where K_n is conjugate to H, and each $K_i < K_{i+1}$ is of one of the following two types:

1. An extension of the form $K_i \lhd K_{i+1}$ where K_{i+1}/K_i is a p-group.

2. There is an infinite sequence of extensions

$$K_i \lhd P_1 \lhd P_2 \lhd P_3 \lhd \cdots$$

   where each quotient group P_{j+1}/P_j is a p-group, $\overline{\cup_j P_j} = K_{i+1}$, and the sequence of the (P_j) converges to (K_{i+1}) in ΦG.

For inclusions of subgroups of type (1), if the degree of $f^{K_{i+1}}$ is prime to p then the degree of f^{K_i} is prime to p by Step 1.

We prove the same statement for inclusions of subgroups of type (2). Assume that the degree of $f^{K_{i+1}}$ is prime to p. The map $f : S^0_G \to Z$ is realized by an unstable map $\tilde{f} : S^V \to X$, where X is a homotopy representation. In particular both S^V and X have finite orbit types [13]. It follows that $\tilde{f}^{K_{i+1}} = \tilde{f}^{P_m}$ for some m. Hence $\deg(f^{P_m}) = \deg(f^{K_{i+1}})$ is prime to p. Repeated use of the result in Step 1 gives that the degree of f^{K_i} is also prime to p.

Finally we conclude that if the degree of f^L is prime to p then the degree of f^K is also prime to p. □

Corollary 2.2.8. *Let Z be a stable homotopy representation with non-negative dimension function $\dim(Z) \geq 0$, such that $\mathcal{E}(Z)$ is a cofamily. Then for every $(H) \in \mathcal{E}(Z)$ and every prime number p there is a G-map $f : S^0_G \to Z$ such that $deg(f^H)$ is relatively prime to p.*

Proof. There is a subgroup K containing H, with $|WK|$ relatively prime to p, whose conjugacy class is in $\Phi(H,p)$. Hence the result follows from Propositions 2.1.4 and 2.2.7. □

The following description of the Burnside ring localized at its prime ideals is from [14, V.5.1].

Proposition 2.2.9. *The localization of $A(G)$ at a minimal prime ideal $q(H,0)$ is $\mathbb{Q}$. The localization of $A(G)$ at a maximal ideal $q(H,p)$ is $(A(G)/I(H,p))_{(p)}$ where $I(H,p) = \cap q(K,0)$ and the intersection is over $(K) \in \Phi(H,p)$.*

More generally, we obtain the following two propositions by mimicking the proof of [14, V.5.1].

Proposition 2.2.10. *Let $\mathcal{E}(Z)$ be a closed subspace in ΦG. The localization of $A(G)/I(Z)$ at a minimal prime ideal $q(H,0)$ is $\mathbb{Q}$. The localization of $A(G)$ at a maximal ideal $q(H,p)$ of $A(G)/I(Z)$ is $(A(G)/J(Z;H,p))_{(p)}$ where*

$$J(Z;H,p) = \{\alpha \in A(G) \mid deg\ (\alpha^L) = 0 \quad for \quad (L) \in \mathcal{E}(Z) \cap \Phi(H,p)\}.$$

Proof. The kernel of the localization map is contained in $J(Z;H,p)/I(Z)$, since the degree function of any element in $A(G) - q(H,p)$ is never zero on $\mathcal{E}(Z) \cap \Phi(H,p)$.

We show that $J(Z;H,p)$ is zero localized at $q(H,p)$. Assume $\alpha \in A(G)$ is such that deg $(\alpha^L) = 0$ for all $(L) \in \mathcal{E}(Z) \cap \Phi(H,p)$. Since the absolute value of the degree function is locally constant, there is an open set U containing $\mathcal{E}(Z) \cap \Phi(H,p)$ such that deg $(\alpha^L) = 0$ for all $(L) \in U$.

For all $(K) \in \mathcal{E}(Z) - \Phi(H,p)$ we have that $q(K,0) \not\subset q(H,p)$ [5, 2.5.7] so there exists a β_K with deg $(\beta_K)^K = 0$ and $\beta_K \notin q(H,p)$. By assumption $\mathcal{E}(Z)$ is a closed subset of ΦG, hence compact. So $\mathcal{E}(Z) \cap (\Phi G - U)$ is compact. We can then find finitely many $(K_1), (K_2), \ldots, (K_n) \in \mathcal{E}(Z) - \Phi(H,p)$ such that the product $\beta = \beta_{K_1}\beta_{K_2}\cdots\beta_{K_n}$ is zero on $\mathcal{E}(Z) \cap (\Phi G - U)$. Since $q(H,p)$ is a prime ideal we have $\beta \notin q(H,p)$. The product $\alpha\beta \in I(Z)$, so

$$J(Z;H,p)/I(Z) \subset \ker(A(G)/I(Z) \to (A(G)/I(Z))_{q(H,p)}).$$

Hence we have an injection

$$(A(G)/J(Z;H,p))_{(p)} \hookrightarrow (A(G)/I(Z))_{q(H,p)}.$$

Since $I(H,p) \subset J(Z;H,p)$ we get that the map is an isomorphism using Proposition 2.2.9. ◻

Proposition 2.2.11. *Let $\mathcal{E}(Z)$ be a cofamily. The localization of $\{S^V, X\}$ at a minimal prime ideal $q(H,0)$ of $A(G)/I(Z)$ is $\mathbb{Q}$. The localization of $\{S^V, X\}$ at a maximal ideal $q(H,p)$ is $(\{S^V, X\}/\tilde{J}(H,p))_{(p)}$ where*

$$\tilde{J}(Z;H,p) = \{h \in \{S^V, X\} \mid deg\ (h^K) = 0 \ \ for\ \ (K) \in \Phi(H,p)\}$$

Proof. For every $f \in \{S^V, X\}$ the absolute value of the degree map

$$|\text{deg } (f)| : \mathcal{E}(Z) \to \mathbb{N}$$

is continuous. Using Proposition 2.2.7 we obtain, as in the previous proposition, that

$$\tilde{J}(Z,H,p) = \ker(\{S^V, X\} \to \{S^V, X\}_{q(H,p)})$$

so $\{S^V, X\}/\tilde{J}(Z; H, p) \to \{S^V, X\}_{q(H,p)}$ is an injective homomorphism. Since $\{S^V, X\} \otimes I(H, p) \subset \tilde{J}(Z; H, p)$ we have that

$$\{S^V, X\}/\tilde{J}(Z; H, p) \cong \{S^V, X\}/\tilde{J}(Z; H, p) \otimes_{A(G)} A(G)/I(H, p).$$

Hence

$$(\{S^V, X\}/\tilde{J}(Z; H, p))_{(p)} \cong \{S^V, X\}/\tilde{J}(Z; H, p) \otimes_{A(G)} A(G)_{q(H,p)}$$

maps surjectively to

$$\{S^V, X\}_{q(H,p)} \cong \{S^V, X\} \otimes_{A(G)} A(G)_{q(H,p)}.$$

We conclude that $(\{S^V, X\}/\tilde{J}(Z; H, p))_{(p)} \hookrightarrow \{S^V, X\}_{q(H,p)}$ is an isomorphism. $\square$

The propositions above and Theorem 2.2.4 give the following commutative diagram with injective vertical maps.

$$\begin{array}{ccc} (A(G)/I(Z))_{q(H,p)} & \xrightarrow{f_*} & \{S^V, X\}_{q(H,p)} \\ {\scriptstyle \deg}\downarrow & & \downarrow{\scriptstyle \deg} \\ \prod_{\mathcal{E}(Z) \cap \Phi(H,p)} \mathbb{Z}_{(p)} & \xrightarrow{\deg\,(f)} & \prod_{\mathcal{E}(Z) \cap \Phi(H,p)} \mathbb{Z}_{(p)} \end{array}$$

If $|WH|$ is relatively prime to p, and if $f : S^V \to X$ satisfies $\deg\,(f^H) \not\equiv 0 \bmod p$, then by Proposition 2.2.7 we get that $\deg\,(f^K) \not\equiv 0 \bmod p$ for all $(K) \in \mathcal{E}(Z) \cap \Phi(H, p)$. Hence the lower horizontal map $\deg\,(f)_{(p)}$ is an isomorphism. In particular this implies that

$$f_* : (A(G)/I(Z))_{q(H,p)} \to \{S^V, X\}_{q(H,p)}$$

is injective. We will prove surjectivity of f_* by showing that the cokernels of the vertical degree homomorphisms in the diagram above are isomorphic.

More precisely, we will use unstable approximations where the degree functions inject into a finite product of the integers. The map between any pair of these cokernels is surjective, and we show that the two cokernels have the same finite order, hence the map between them is an isomorphism. We then stabilize to show that f_* is an isomorphism localized at $q(H,p)$.

Let U be any complex G-representation. For convenience we denote the ring $[S^{V\oplus U}, S^{V\oplus U}]$ by R_U, and the R_U-module $[S^{V\oplus U}, X \wedge S^U]$ by M_U. We have that

$$\mathrm{colim}_U R_U \cong A(G).$$

and

$$\mathrm{colim}_U M_U \cong \{S^V, X\} \cong \pi_0^G(Z).$$

Definition 2.2.12. Let $J_U \subset R_U$ be the ideal of homotopy classes of G-maps $\alpha : S^{V\oplus U} \to S^{V\oplus U}$ such that deg $(\alpha^K) = 0$ for all $(K) \in \Phi(H,p) \cap \mathcal{E}(Z)$. Let $\tilde{J}_U \subset M_U$ be the submodule of homotopy classes of G-maps $\beta : S^{V\oplus U} \to X \wedge S^U$ such that deg $(\beta^K) = 0$ for all $(K) \in \Phi(H,p)$.

These ideals are defined so that the maps below are injections

$$R_U/J_U \hookrightarrow A(G)/J(Z;H,p)$$

$$M_U/\tilde{J}_U \hookrightarrow \{S^V, X\}/\tilde{J}(Z;H,p).$$

Lemma 2.2.13. *The canonical maps*

$$colim_U(R_U/J_U)_{(p)} \to (A(G)/I(Z))_{q(H,p)} \cong (A(G)/J(Z;H,p))_{(p)}$$

$$colim_U(M_U/\tilde{J}_U)_{(p)} \to \{S^V, X\}_{q(H,p)} \cong (\{S^V, X\}/\tilde{J}(Z;H,p))_{(p)}$$

are isomorphisms.

Proof. We prove the first isomorphism. The proof of the second is similar. Colimits commute with localization, so it is enough to prove the isomorphism before localizing at p. We have injections

$$[S^{V\oplus U}, S^{V\oplus U}]/J_U \hookrightarrow A(G)/J(Z;H,p).$$

Since $\mathrm{colim}_U[S^{V\oplus U}, S^{V\oplus U}] \to A(G)$ is an isomorphism it is easy to see that

$$\mathrm{colim}_U[S^{V\oplus U}, S^{V\oplus U}]/J_U \to A(G)/J(Z; H, p)$$

is an isomorphism. □

The congruence relations in Theorem 2.2.4 imply that the degree maps give rise to injective homomorphisms

$$R_U \longrightarrow \prod\nolimits_{\mathrm{iso}(V\oplus U)} \mathbb{Z} \quad \text{and} \quad M_U \longrightarrow \prod\nolimits_{\mathrm{iso}(V\oplus U)\cap\mathcal{E}(Z)} \mathbb{Z}.$$

To describe the order of the cokernels of the maps above we introduce some notation modeled on Proposition 2.2.4. Let $R \overset{\phi}{\hookrightarrow} \prod_{i=1}^{n} \mathbb{Z}$ be a subgroup with the following property: The values for the k-th coordinates of elements in R with the l-th coordinates all zero for $l > k$ are exactly $w_k\mathbb{Z}$, where w_k is some positive integer.

Filter $P = \prod_{i=1}^{n} \mathbb{Z}$ by letting $P_k = \prod_{i=1}^{k} \mathbb{Z} \times 0$. The associated graded group is $\oplus_{i=1}^{n}\mathbb{Z}$. Let $R_k = \phi^{-1}(P_k)$. This gives a filtration of R. Let $C_k = \mathrm{coker}(P_k \to R_k)$ and $C_n = C$. We have that

$$\mathrm{gr}(R) \to \mathrm{gr}(P) \to \mathrm{gr}(C)$$

is a short exact sequence. The k-th term in $\mathrm{gr}(C)$, C_k/C_{k-1}, is the cokernel of $R_k/R_{k-1} \to \mathbb{Z}$. The image of this map is the k-th coordinate of all elements f in R such that $f(i) = 0$ for all $i \geq k+1$. This is by assumption exactly $w_k\mathbb{Z}$. Hence $C_k/C_{k-1} \cong \mathbb{Z}/w_k\mathbb{Z}$.

Definition 2.2.14. Let $\Lambda_U = \mathrm{iso}(V \oplus U) \cap \mathcal{E}(Z) \cap \Phi(H, p)$.

Lemma 2.2.15. *The orders of the cokernels*

$$C_U = \mathit{coker}(R_U/J_U \to \prod\nolimits_{\Lambda_U} \mathbb{Z}) \quad \text{and}$$

$$\tilde{C}_U = \mathit{coker}(M_U/\tilde{J}_U \to \prod\nolimits_{\Lambda_U} \mathbb{Z})$$

are equal.

Proof. We use the above remark for

$$R_U \to \prod_{\mathrm{iso}(V \oplus U)} \mathbb{Z}.$$

Order the isotropy classes of $\mathrm{iso}(V \oplus U)$ so that $i < j$ implies that $(H_i) < (H_j)$. By Theorem 2.2.4 we get that the k-th term of the associated graded of

$$C'_U = \text{coker } (R_U \to \prod_{\mathrm{iso}(V \oplus U)} \mathbb{Z})$$

is $\mathbb{Z}/|WH_k|\mathbb{Z}$.

In the following diagram the horizontal sequences are exact, and the two leftmost vertical sequences are exact. It then follows that the rightmost vertical sequence is also exact.

$$\begin{array}{ccccc}
R_U/J_U & \longrightarrow & \prod_{\Lambda_U} \mathbb{Z} & \longrightarrow & C_U \\
\uparrow & & \uparrow & & \uparrow \\
R_U & \longrightarrow & \prod_{\mathrm{iso}(V \oplus U)} \mathbb{Z} & \longrightarrow & C'_U \\
\uparrow & & \uparrow & & \uparrow \\
J_U & \longrightarrow & \prod_{\mathrm{iso}(V \oplus U) - \Lambda_U} \mathbb{Z} & \longrightarrow & (\prod_{\mathrm{iso}(V \oplus U) - \Lambda_U} \mathbb{Z})/J_U
\end{array}$$

Hence the associated graded of the cokernel C_U is the cokernel of

$$\bigoplus_{\mathrm{iso}(V \oplus U) - \Lambda_U} \mathbb{Z} \to \mathrm{gr}(C'_U)$$

which is

$$\bigoplus_{(H) \in \Lambda_U} \mathbb{Z}/|WH|\mathbb{Z}.$$

The same argument applied to

$$\text{coker } (M_U \to \prod_{\mathrm{iso}(V \oplus U) \cap \mathcal{E}(Z)} \mathbb{Z})$$

and the ideal $\tilde{J}_U$ gives that the associated graded of the cokernel $\tilde{C}_U$ of

$$M_U/\tilde{J}_U \to \prod_{\Lambda_U} \mathbb{Z}$$

is

$$\bigoplus_{(H)\in\Lambda_U} \mathbb{Z}/|WH|\mathbb{Z}.$$

Hence the orders of both C_U and $\tilde{C}_U$ are $\prod_{(H)\in\Lambda_U} |WH|$. □

Proposition 2.2.16. *If $f : S^V \to X$ is a map such that deg $(f^H) \not\equiv 0$ mod p, and $|WH|$ is prime to p, then the map*

$$f_* : (A(G)/I(Z))_{q(H,p)} \to \{S^V, X\}_{q(H,p)}$$

is an isomorphism.

Proof. By Proposition 2.2.13 it is enough to prove that for every U the map

$$f_* : (R_U/J_U)_{(p)} \to (M_U/\tilde{J}_U)_{(p)}$$

is an isomorphism. Consider the following map of short exact sequences

$$\begin{array}{ccccc} (R_U/J_U)_{(p)} & \longrightarrow & (\prod_{\Lambda_U} \mathbb{Z})_{(p)} & \longrightarrow & C_{(p)} \\ \downarrow f_* & & \downarrow \deg(f) & & \downarrow \deg(f) \\ (M_U/\tilde{J}_U)_{(p)} & \longrightarrow & (\prod_{\Lambda_U} \mathbb{Z})_{(p)} & \longrightarrow & \tilde{C}_{(p)}. \end{array}$$

By Proposition 2.2.7 the values of deg (f) on Λ_U are all relatively prime to p, so we get that deg $(f) : (\prod_{\Lambda_U} \mathbb{Z})_{(p)} \to (\prod_{\Lambda_U} \mathbb{Z})_{(p)}$ is an isomorphism. Hence the map $C_{(p)} \to \tilde{C}_{(p)}$ is surjective. Since C is a finite group $|C_{(p)}|$ is the p-part of $|C|$. By the previous lemma the orders $|C_{(p)}|$ and $|\tilde{C}_{(p)}|$ are equal. Hence the surjective map $C_{(p)} \to \tilde{C}_{(p)}$ must also be injective. The five-lemma implies that f_* is an isomorphism. □

Proposition 2.2.17. *Let Z be a stable homotopy representation with $\mathcal{E}(Z)$ a cofamily and* $\dim Z \geq 0$. *Then $\pi_0(Z)$ is a finitely generated $A(G)$-module.*

Proof. For any $(H) \in \mathcal{E}(Z)$ choose a prime number p relatively prime to $|WH|$. There is, by Proposition 2.1.4, a map $f : S^0_G \to Z$ such that $\deg(f^H) \not\equiv 0 \mod p$. Since $|\deg f|$ is locally constant, there is an open neighborhood U_H of (H), such that $|\deg f^H| = |\deg f^K|$ for all $(K) \in U_H$. Hence by Proposition 2.2.16 we get that the induced map

$$f_* : (A(G)/I(Z))_{q(K,l)} \to \pi_0(Z)_{q(K,l)}$$

is an isomorphism whenever l is a prime that does not divide $|\deg f^H|$ and $(K) \in U_H$ with $|WK|$ prime to l.

For each of the primes p' that divides $|\deg f^H|$ there is by corollary 2.2.8 a map $g : S^G_0 \to Z$ such that $\deg(g^H) \not\equiv 0 \mod p'$, hence by Proposition 2.2.16, it induces an isomorphism

$$g_* : (A(G)/I(Z))_{q(K,p')} \to \pi_0(Z)_{q(K,p')}$$

for all (K) with $|WK|$ relatively prime to p' in some neighborhood of (H) in $\mathcal{E}(Z)$.

Hence for every (H) in $\mathcal{E}(Z)$ there is an open neighborhood V_H of (H) in $\mathcal{E}(Z)$ and a finite set of maps from S^0_G to Z, such that at least one of the maps induces an isomorphism localized at $q(K,p)$ for any prime p and any $(K) \in V_H$ with $|WK|$ prime to p.

The subset $\mathcal{E}(Z)$ is a compact space by Lemma 2.2.2. Since $\mathcal{E}(Z)$ is a cofamily, any maximal ideal is of the form $q(H,p)$ where $(H) \in \mathcal{E}(Z)$ and $|WH|$ is prime to p, it follows, by a compactness argument, that there are finitely many maps

$$f_1, f_2, \ldots, f_k : S^0_G \to Z$$

such that for any maximal ideal $q(H,p)$ of $A(G)/I(Z)$, at least one of them induces an isomorphism on homotopy groups localized at $q(H,p)$. So the map

$$\oplus_{i=1}^k f_i : \oplus_{i=1}^k A(G)/I(Z) \to \pi_0^G(Z)$$

is surjective localized at all the maximal ideals of $A(G)/I(Z)$, hence is a surjective homomorphism. This proves that $\pi_0^G(Z)$ is a finitely generated $A(G)/I(Z)$-module.

□

From [1, II.5] Theorem 1 and 3 we know that an R-module M is invertible if and only if

1. For all maximal ideals $\mathfrak{m}$ of R we have that $M_{\mathfrak{m}} \cong R_{\mathfrak{m}}$

2. M is a finitely generated R-module.

The last two Propositions now imply Theorem 2.1.5.

Remark 2.2.18. When Z is a stable homotopy representation with $\dim(Z) = 0$ it turns out that $\mathcal{E}(Z) = \Phi G$ [5, 10.16] [8]. It also follows from the arguments in the above references that whenever the subset $\mathcal{E}'(Z) = \{ (K) \in \Phi G \mid X^K \simeq S^{V^K} \}$ is a cofamily, then $\mathcal{E}'(Z) = \mathcal{E}(Z)$.

The group of isomorphism classes of stable homotopy representations with dimension function identically zero is isomorphic to the group of isomorphism classes of invertible $A(G)$-modules. The isomorphism is given by sending X to $\pi_0^G(X)$. Moreover, if X is a stable homotopy representation with dimension function identically zero, then

$$\pi_0^G(Y) \otimes_{A(G)} \pi_0^G(X) \cong \pi_0^G(Y \wedge X)$$

for any G-spectrum Y [8]. Let Z be any stable homotopy representation with non-negative dimension function such that $\mathcal{E}(Z)$ is a cofamily. Then by Theorem 2.1.3 there is an invertible $A(G)$-module P such that $\pi_0^G(Z) \cong A(G)/I(Z) \otimes_{A(G)} P$. Let P' be any invertible $A(G)$-module. There is a stable homotopy representation X with identically zero dimension function such that $\pi_0^G(X) \cong P^{-1} \otimes_{A(G)} P'$ where P^{-1} is an inverse of P. The stable homotopy representation $Z \wedge X$ has the same dimension function as Z and

$$\pi_0^G(Z \wedge X) \cong A(G)/I(Z) \otimes_{A(G)} P'.$$

In particular we can choose a stable homotopy representation Z' with $\dim(Z') = \dim(Z)$ such that

$$\pi_0^G(Z') \cong A(G)/I(Z).$$

2.3 Invertible $A(G)/I(Z)$-modules

In this section we prove Proposition 2.1.6.

Proposition 2.3.1. *Let G be a compact Lie group, and let $\mathcal{E}$ be any open-closed subspace of $\Phi(G)$. Then*

$$A(G)/I(\mathcal{E}) \otimes - : Pic(A(G)) \to Pic(A(G)/I(\mathcal{E}))$$

is surjective.

Proof. Let C denote the ring of continuous functions from ΦG to the integers, and $C(\mathcal{E})$ the ring of continuous functions from $\mathcal{E}$ to the integers. T. tom Dieck proved that we have a diagram

$$\begin{array}{ccccc} (C/|G|C)^{\times} & \longrightarrow & \mathrm{Pic}(A(G)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ (C(\mathcal{E})/|G|C(\mathcal{E}))^{\times} & \longrightarrow & \mathrm{Pic}(A(G)/I(\mathcal{E})) & \longrightarrow & 0 \end{array}$$

where the horizontal maps are surjective [5, 10.3.8]. The left vertical map is surjective by the assumption on $\mathcal{E}$. Hence $\mathrm{Pic}(A(G)) \to \mathrm{Pic}(A(G)/I)$ is also surjective. □

The Lemmas 2.2.1 and 2.2.2 now imply Proposition 2.1.6.

2.4 The Rational Stems

In this section we denote the order of a compact Lie group G by g.

Lemma 2.4.1. *If p is relatively prime to g then $A(G)_{q(H,p)} \cong \mathbb{Z}_p$.*

Proof. For any $(K) \in \Phi G$ there is a unique conjugacy class (H) with $|WH|$ prime to p and $q(H,p) = q(K,p)$ [5, 5.7.2]. By assumption any $|WH|$ is relatively prime to p. Hence the ideal $I(H,p)$ is $q(H,0)$, and the proof is completed by Proposition 2.2.9. □

The prime ideals of the ring $A(G)/I(Z) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}]$ are exactly the $q(H,0)$ and $q(H,p)$ for any prime number p that does not divide $|G|$ and $(H) \in \overline{\mathcal{E}(Z)}$. By lemma 2.4.1 we get that the localization of $A(G)\otimes_{\mathbb{Z}}\mathbb{Z}[g^{-1}]$ at any maximal ideal $q(H,p)$ is

$$(A(G)\otimes_{\mathbb{Z}}\mathbb{Z}[g^{-1}])_{q(H,p)} \cong \mathbb{Z}_{(p)}.$$

Similarly for $A(G)/I(Z)$ and $\pi_0^G(Z)$ using Propositions 2.2.10 and 2.2.11.

Proposition 2.4.2. *Let G be a compact Lie group. Let Z a stable homotopy representation with dim$(Z) \geq 0$ and $\mathcal{E}(Z)$ a closed subset of ΦG. Then we have an isomorphism*

$$\pi_0^G(Z) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}] \cong (A(G)/I(Z)) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}]$$

of $A(G)$-modules.

Proof. We prove, using Theorem 2.2.4 and induction on orbit types, that there is a map $f : S^V \to X$ such that for any conjugacy class $(H) \in \mathcal{E}(Z)$ and any prime number p that does not divide $|G|$, we have that the degree of f^H is prime to p.

Order the orbit types of S^V that are in $\mathcal{E}(Z)$ such that $(H_i) > (H_j)$ implies that $i < j$. For a given j assume there is a map $f_{j-1} : S^V \to X$ such that $\deg(f_{j-1}^{H_i})$ is relative prime to $|G|$ for all $i < j$. If $|WH_j|$ is infinite let $f_{j-1} = f_j$. If $|WH_j|$ is finite we use Theorem 2.2.4 to alter the map $|WH_j|f_{j-1}$ to obtain a new map f_j, such that the $\deg(f_j^{H_i}) = |WH_j| \deg(f_{j-1}^{H_i})$ for all $i < j$ and $\deg(f_j^{H_j}) = |WH_j|$. This new map f_j has the required property for all $i \leq j$. To start the induction choose a map with $\deg(f^G) = 1$ [5, 10.2.5].

The induced map $f_* : A(G)/I(Z) \to \pi_0^G(Z)$ localized at any maximal ideal $q(H,p)$ of $A(G)/I(Z) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}]$ is

$$\deg(f^H) : \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}.$$

This is an isomorphism, since the degree of f^H is relatively prime to any prime number that does not divide $|G|$. Hence the map f induces the required isomorphism. $\square$

We can say more for finite groups. The following Theorem is from [12, 3.6], it is also implicit in [14, Chap.5] and [5, p. 226].

Theorem 2.4.3. *Let G be a finite group of order g, and let Y be a retract of a finite G-CW complex. Then*

$$\pi_0^G(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}] \cong \prod_{(H)} \pi_0(Y^H)^{WH} \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}].$$

In particular for any stable homotopy representation Z we have that

$$\pi_0^G(Z) \otimes_{A(G)} A(G)/I(Z) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}] \cong A(G)/I(Z) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}].$$

We have an isomorphism of rings

$$A(G)/I(Z) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}] \cong \textstyle\prod_{(H)\in\mathcal{E}(Z)} \mathbb{Z}[g^{-1}].$$

For any invertible $A(G)$-modules P we have that

$$P \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}] \cong A(G) \otimes_{\mathbb{Z}} \mathbb{Z}[g^{-1}].$$

2.5 More on invertible $A(G)/I$-modules

In this section we prove that for a finite group G and any $A(G)$-ideal I the homomorphism

$$A(G)/I \otimes - : \mathrm{Pic}(A(G)) \to \mathrm{Pic}(A(G)/I)$$

given by tensoring an representative for an isomorphism class of invertible $A(G)/I$-modules with $A(G)/I$ is surjective.

For clarity of the argument we prove a more general result.

Lemma 2.5.1. *Let X be a Noetherian scheme of dimension 1, and let $j : V \rightarrowtail X$ be a closed subscheme. If $\mathcal{O}_X^\times \to j_*\mathcal{O}_V^\times$ is epic as a map of sheaves of abelian groups, then $Pic(X) \to Pic(V)$ is surjetive.*

Proof. Let K be the kernel of $\mathcal{O}_X^\times \to j_*\mathcal{O}_V^\times$. From the long exact sequence in cohomology

$$\mathrm{H}^1(X;\mathcal{O}_X^\times) \to \mathrm{H}^1(X;j_*\mathcal{O}_V^\times) \to \mathrm{H}^2(X;K)$$

is exact. Since X is a Noetherian scheme of dimension 1, the Grothendieck vanishing theorem gives that $\mathrm{H}^2(X;K) = 0$ [9, 3.6.5]. Hence the homomorphism

$$\mathrm{Pic}(X) \cong \mathrm{H}^1(X;\mathcal{O}_X^\times) \to \mathrm{H}^1(X;j_*\mathcal{O}_V^\times) \cong \mathrm{H}^1(V;\mathcal{O}_V^\times) \cong \mathrm{Pic}(V)$$

is surjective. □

Lemma 2.5.2. *Let X be a scheme. The canonical map $\lambda_p : (\mathcal{O}^\times)_p \to (\mathcal{O}_p)^\times$ is an isomorphism for all points $p \in X$.*

Proof. The map λ_p composed with the inclusion $(\mathcal{O}_p)^\times \to (\mathcal{O}_p)$ is the localization at the point p of the inclusion $\mathcal{O}^\times \to \mathcal{O}$, hence is itself injective. We conclude that λ_p must be injective.

To show surjectivity, let $y \in (\mathcal{O}_p)^\times$. There exists an open set U containing p and elements $\tilde{y}$ and $\tilde{z}$ in $\mathcal{O}(U)$ such that $\tilde{y}$ restricts to y in $\mathcal{O}_p$, and $\tilde{y}\tilde{z}$ restricts to 1 in $\mathcal{O}_p$. Hence there is an open set V containing p in U such that $\tilde{y}\tilde{z}|V$ is 1 in $\mathcal{O}(V)$. We conclude that y comes from an element in $\mathcal{O}^\times(V)$ hence from $(\mathcal{O}^\times)_p$. □

In particular the map $\mathcal{O}_X^\times \to j_*\mathcal{O}_V^\times$ is epic if and only if for all points $p \in V$ the homomorphism $(\mathcal{O}_{X,p})^\times \to (\mathcal{O}_{V,p})^\times$ is surjective.

Proposition 2.5.3. *Let R be a Noetherian ring of dimension 1, and let I be an ideal of R such that for all prime ideals $\mathfrak{p}$ of R/I the induced map of units of the quotient map*

$$R_{\mathfrak{p}}^\times \to (R_{\mathfrak{p}}/I_{\mathfrak{p}})^\times$$

is surjective. Then the homomorphism

$$R/I \otimes - : Pic(R) \to Pic(R/I)$$

is surjective.

Proof. Let V be the closed subscheme $\operatorname{spec}(R/I)$ and apply Proposition 2.5.1. It is easy to identify the homomorphism as given by the tensor product with R/I. $\square$

Lemma 2.5.4. *Let G be a compact Lie group, and let I be any ideal in the Burnside ring $A(G)$. Then for all prime ideals q of $A(G)$ the map*

$$(A(G)_q)^\times \to ((A(G)/I)_q)^\times$$

is surjective.

Proof. If $q \not\supseteq I$ then the statement is trivially true. Assume that $q(H,p) \supseteq I$, where H is any subgroup of G and p is 0 or a prime number. We then have for all $i_\bullet \in I$, via the degree function, that $i_H \equiv 0 \bmod p$. The elements of $A(G)$ are identified with elements in a direct product of the integers indexed over the conjugacy classes of closed subgroups of G. Let $m_\bullet/s_\bullet \in A(G)_{q(H,p)}$ be an element such that $[m_\bullet/s_\bullet] \in ((A(G)/I)_{q(H,p)})^\times$. Then there is an element $n_\bullet/t_\bullet \in A(G)_{q(H,p)}$ such that $m_\bullet n_\bullet/s_\bullet t_\bullet = 1$ in $((A(G)/I)^\times_{q(H,p)}$. Hence there is a $k_\bullet \notin q(H,p)$ and an $i_\bullet \in I$ such that

$$k_\bullet = (k_\bullet m_\bullet n_\bullet/s_\bullet t_\bullet) + i_\bullet$$

Since $k_\bullet, s_\bullet, t_\bullet \notin q(H,p)$ and $I \subseteq q(H,p)$ we get that

$$0 \not\equiv k_H s_H t_H \equiv k_H m_H n_H \bmod p.$$

This implies that both $m_\bullet$ and $n_\bullet$ are not in $q(H,p)$ hence invertible in $A(G)_{q(H,p)}$. $\square$

The Burnside ring $A(G)$ of a finite group G is a Noetherian ring of dimension 1. As a consequence we get

Proposition 2.5.5. *Let G be a finite group, and let I be any ideal in the Burnside ring $A(G)$. Then the following homomorphism is surjective*

$$A(G)/I \otimes - : Pic(A(G)) \to Pic(A(G)/I).$$

When G is a compact Lie group $A(G)$ is a ring of dimension 1 but it is typically not Noetherian. It is unknown if the vanishing theorem still applies in this case [7].

Part II

Picard Groups Of Derived Categories

CHAPTER 3

PICARD GROUPS OF DERIVED CATEGORIES

3.1 The derived tensor product and K-flat resolutions

We recall the following definition from [17, 5.1].

Definition 3.1.1. A complex $F^\bullet$ is K-flat if for all acyclic complexes $A^\bullet$ the complex $F^\bullet \otimes_{\mathcal{O}} A^\bullet$ is acyclic. A K-flat resolution of a complex $M^\bullet$ is a cohomology isomorphism $F^\bullet \to M^\bullet$ where $F^\bullet$ is a K-flat complex.

For an object X in the site $\mathcal{C}$ let X also denote the sheaf of sets in $\mathcal{E}$ represented by the object X. Let $j_X : \mathcal{E}|X \to \mathcal{E}$ be the localization map [10, 5.2]. The $\mathcal{O}$-module $j_{X!}j_X^*\mathcal{O}$ is called the free $\mathcal{O}$-module generated by X, and we denote it by $\mathcal{O}_X$ [10, 11.3.1]. There is an adjunction [10, 11.3.3]

$$\hom_{\mathcal{M}}(\mathcal{O}_X, M) \cong \hom_{\mathcal{E}}(X, M) = M(X).$$

For any point p of $\mathcal{E}$ there is a canonical isomorphism of stalks

$$(\mathcal{O}_X)_p \cong \oplus_{X_p} \mathcal{O}_p$$

where the sum is over the set X_p [10, 11.3.5]. Hence $(\mathcal{O}_X)_p$ is a flat $\mathcal{O}_p$-module for all points p of $\mathcal{E}$. Since $\mathcal{E}$ has enough points it follows that $\mathcal{O}_X$ is a flat $\mathcal{O}$-module.

For each element $m \in M(X)$ the adjunction gives a unique morphism $\mathcal{O}_X \to M$ of $\mathcal{O}$-modules. Since the site $\mathcal{C}$ is small we can construct the following epic map of $\mathcal{O}$-modules

$$F(M) = \oplus_{X \in \mathcal{C}} \oplus_{m \in M(X) \setminus \{0\}} \mathcal{O}_X \to M.$$

When $M = 0$ we set $F(M) = 0$. This gives a canonical resolution of any $\mathcal{O}$-module M

$$\cdots \xrightarrow{d^{-3}} F(\ker(d^{-1})) \xrightarrow{d^{-2}} F(\ker(d^{0})) \xrightarrow{d^{-1}} F(M) \xrightarrow{d^{0}} M \to 0.$$

Let us denote this resolution $\mathcal{F}^\bullet(M)$. For any map $g : M \to N$ there is a canonical map $F(g) : F(M) \to F(N)$ such that $d^0 \circ F(g) = g \circ d^0$. This gives a canonical map $\mathcal{F}^\bullet(g) : \mathcal{F}^\bullet(M) \to \mathcal{F}^\bullet(N)$ of the resolutions. Hence any bounded above cochain complex of $\mathcal{O}$-modules

$$\cdots \to M^{n-2} \to M^{n-1} \to M^n \to 0$$

has a resolution which is (the totalization of)

$$\cdots \to \mathcal{F}^\bullet(M^{n-2}) \to \mathcal{F}^\bullet(M^{n-1}) \to \mathcal{F}^\bullet(M^n) \to 0.$$

We call this resolution the standard resolution. For each point p of $\mathcal{E}$ the complex $\mathcal{F}^\bullet(M)_p$ is a complex of free $\mathcal{O}_p$-modules. Hence the standard resolution localized at a point p is a bounded above complex of free $\mathcal{O}_p$-modules, which implies that it is a K-flat complex of $\mathcal{O}_p$-modules [17, 3.2, 5.8]. Since $\mathcal{E}$ has enough points it follows that the standard resolution is a K-flat complex of $\mathcal{O}$-modules. Hence all bounded above cochain complexes have K-flat resolutions.

For an $\mathcal{O}$-module M and an integer n we denote the complex which is M in degree n and trivial in all other degrees by $M[n]$. The map $M \mapsto M[0]$ defines a monomorphism $\mathrm{Pic}(\mathcal{M}) \to \mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$. Let $\tau_{\leq n} M^\bullet$ denote the sub-complex of $M^\bullet$ which is 0 in degrees above n, the kernel of $d : M^n \to M^{n+1}$ in degree n, and agrees with $M^\bullet$ in degrees below n. The map $\tau_{\leq n} M^\bullet \to M^\bullet$ induces an isomorphism on cohomology groups in degrees less than or equal to n. The following lemma is from [17, 3.3].

Lemma 3.1.2. *Let $M^\bullet$ be a cochain complex of $\mathcal{O}$-modules. Fix an integer k. There exists a sequence*

$$F_k^\bullet \to F_{k+1}^\bullet \to F_{k+2}^\bullet \to \cdots$$

of complexes of K-flat $\mathcal{O}$-modules mapping into $M^\bullet$ such that $F_n^q = 0$ for all $q > n$ and $\mu_n : F_n^\bullet \to \tau_{\leq n} M^\bullet$ is a cohomology isomorphism. Let $F^\bullet = colim(F_n^\bullet)$. In each degree $F^\bullet$ is a sum of flat $\mathcal{O}$-modules of the form $\mathcal{O}_X$ for $X \in \mathcal{C}$, and $F^\bullet$ is a K-flat complex. The map $F^\bullet \to M^\bullet$ is a K-flat resolution of $M^\bullet$.

Proof. For the given k, let $F_k^\bullet \to \tau_{\leq k} M^\bullet$ be the standard resolution. We now construct $F_n^\bullet$ inductively. Assume that $\mu_{n-1} : F_{n-1}^\bullet \to \tau_{\leq n-1} M^\bullet$ is given. Let μ'_{n-1} denote the composite of μ_{n-1} with the inclusion $\tau_{\leq n-1} M^\bullet \to \tau_{\leq n} M^\bullet$. Construct the following diagram:

$$\begin{array}{ccccccccc}
F_{n-1}^\bullet & \xrightarrow{\mu'_{n-1}} & \tau_{\leq n} M^\bullet & \longrightarrow & C_{\mu'_{n-1}} & \longrightarrow & F_{n-1}^\bullet[1] & \xrightarrow{\mu'_{n-1}[1]} & \tau_{\leq n} M^\bullet[1] \\
 & & & & \simeq \uparrow & & \| & & \uparrow \\
 & & & & Q^\bullet & \xrightarrow{g} & F_{n-1}^\bullet[1] & \longrightarrow & C_g .
\end{array}$$

The cone $C_{\mu'_{n-1}}$ is a complex which is 0 in degrees above n, and we let $Q^\bullet \to C_{\mu'_{n-1}}$ be the standard resolution. In particular Q^i is 0 for $i > n$.

Define $F_n^\bullet$ to be $C_g[-1]$ and let μ_n be the evident cohomology isomorphism $C_g[-1] \to \tau_{\leq n} M^\bullet$. By construction $F_n^i = 0$ for $i > n$, and since both $Q^\bullet$ and $F_{n-1}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules $F_n^\bullet$ is a K-flat complex of $\mathcal{O}$-modules. Note that μ_{n-1} is the composite map

$$F_{n-1}^\bullet \to F_n^\bullet \overset{\mu_n}{\to} \tau_{\leq n} M^\bullet .$$

Since cohomology commutes with filtered colimits $F^\bullet = \text{colim}(F_n^\bullet) \to M^\bullet$ is a cohomology isomorphism and, since each $F_n^\bullet$ is K-flat, $F^\bullet$ is also K-flat. □

Remark 3.1.3. A ring R (resp. an R-module) is a ringed space (resp. module over the ringed space) when the site is the category with only one object and one morphism.

3.2 The Picard group of the derived category of R-modules

Let R be a commutative unital ring.

Lemma 3.2.1. *Let $M^\bullet$ be an invertible object in $\mathcal{D}_R$. Then there exists an integer m such that $H^q(M^\bullet) = 0$ for $q > m$ and $H^m(M^\bullet) \neq 0$ is a finitely generated R-module.*

Proof. Replace $M^\bullet$ by a K-flat resolution $F^\bullet$ as in Lemma 3.1.2. $R[0]$ is the unit object in $\mathcal{D}_R$. Let $F^\bullet \otimes_R G^\bullet \cong R[0]$ in $\mathcal{D}_R$ where $G^\bullet$ is a K-flat complex.

By choosing a cycle representing the image of 1 under the isomorphism $R \cong \mathrm{H}^0(F^\bullet \otimes_R G^\bullet)$ we get a map of cochain complexes

$$\eta : R[0] \to F^\bullet \otimes_R G^\bullet$$

inducing an isomorphism on cohomology. Since $F^\bullet = \mathrm{colim}(F_n^\bullet)$ the map η factors through $F_n^\bullet \otimes_R G^\bullet$ for some integer n. By tensoring the commutative triangle

$$\begin{array}{ccc} & & F_n^\bullet \otimes_R G^\bullet \\ & \nearrow & \downarrow \\ R[0] & \xrightarrow{\cong} & F^\bullet \otimes_R G^\bullet \end{array}$$

with $F^\bullet$ from the right and using the equivalence $G^\bullet \otimes_R F^\bullet \cong R[0]$ in $\mathcal{D}_R$ we see that $F^\bullet$ is a retract of $F_n^\bullet$ in $\mathcal{D}_R$. Since $F_n^q = 0$ for $q > n$ and $M^\bullet$ has nontrivial cohomology we conclude that there is an integer m such that $\mathrm{H}^q(M^\bullet) = 0$ for $q > m$ and $\mathrm{H}^m(M^\bullet) \neq 0$. In particular $\mu_m : F_m^\bullet \to M^\bullet$ is a cohomology isomorphism.

Next we show that $F_m^\bullet$ is a retract in $\mathcal{D}_R$ of a bounded complex $B^\bullet$ which is a finitely generated R-module in each degree and satisfies $B^q = 0$ for $q > m$. As above there is a map of cochain complexes

$$\eta : R[0] \to F_m^\bullet \otimes_R G^\bullet$$

inducing an isomorphism on cohomology. Let $\eta(1) = \Sigma_{i,j\in\mathbb{Z}} f_{i,j} \otimes_R g_{i,j}$ where $f_{i,j} \in F^i$ and $g_{i,j} \in G^{-i}$ for all i and j, and almost all the $f_{i,j}$ are zero. Let $B^\bullet$ be the subcomplex of $F_m^\bullet$ generated by all the $f_{i,j}$. Since almost all the $f_{i,j}$ are zero, $B^\bullet$ is a bounded subcomplex of $F_m^\bullet$, so $B^q = 0$ for $q > m$, and B^n is a finitely generated R-module for each integer n.

The map η factors through $B^\bullet \otimes_R G^\bullet$. Tensoring the factorization with $F^\bullet_m$ from the right gives the commutative diagram

$$\begin{array}{ccccc} & \nearrow & B^\bullet \otimes_R G^\bullet \otimes_R F^\bullet_m & \xrightarrow{\cong} & B^\bullet \\ & & \downarrow & & \downarrow \\ F^\bullet_m & \xrightarrow{\cong} & F^\bullet_m \otimes_R G^\bullet \otimes_R F^\bullet_m & \xrightarrow{\cong} & F^\bullet_m \, . \end{array}$$

Since both $F^\bullet_m$ and $G^\bullet$ are K-flat all the tensor products are derived tensor products. We get that $F^\bullet_m$, hence $M^\bullet$, is a retract of $B^\bullet$ in $\mathcal{D}_R$. In particular $\mathrm{H}^m(M^\bullet)$ is a retract of $\mathrm{H}^m(B^\bullet) = B^m/d(B^{m-1})$. Hence $\mathrm{H}^m(M^\bullet)$ is a finitely generated R-module. □

It is easy to see that if the cohomology of $M^\bullet$ is concentrated in one degree, say n, then $M^\bullet$ is isomorphic to $\mathrm{H}^n(M^\bullet)[n]$ in $\mathcal{D}_R$. The proof of the next result follows [11, V.3.3].

Proposition 3.2.2. *If R is a local ring then up to isomorphism the invertible objects in $\mathcal{D}_R$ are precisely the $R[n]$ for any integer n.*

Proof. The complex $R[n]$ is invertible for any n since $R[n] \otimes^{\mathrm{L}}_R R[-n] \cong R[0]$.

Let $M^\bullet \otimes^{\mathrm{L}}_R N^\bullet \cong R[0]$ in $\mathcal{D}_R$. Since the complexes $M^\bullet$ and $N^\bullet$ are bounded from above, there is a convergent spectral sequence

$$E_2^{p,q} = \bigoplus_{i+j=q} \mathrm{Tor}^p_R(\mathrm{H}^i(M^\bullet), \mathrm{H}^j(N^\bullet)) \Rightarrow \mathrm{H}^{p+q}(M^\bullet \otimes^{\mathrm{L}}_R N^\bullet)$$

where the grading is so that Tor^p_R is zero for $p > 0$. Since $M^\bullet \otimes^L_R N^\bullet \cong R[0]$ the cohomology group $\mathrm{H}^r(M^\bullet \otimes^{\mathrm{L}}_R N^\bullet)$ is isomorphic to R when $r = 0$ and is zero when $r \neq 0$. If $\mathrm{H}^i(M^\bullet) = 0$ for $i > m$ and $\mathrm{H}^j(N^\bullet) = 0$ for $j > n$, the spectral sequence gives us that the Künneth map

$$\mathrm{H}^m(M^\bullet) \otimes_R \mathrm{H}^n(N^\bullet) \to \mathrm{H}^{m+n}(M^\bullet \otimes^{\mathrm{L}}_R N^\bullet)$$

is an isomorphism. If, further, $\mathrm{H}^m(M^\bullet) \neq 0$ and $\mathrm{H}^n(N^\bullet) \neq 0$, then by Lemma 3.2.1 both modules are finitely generated. One proves by induction on the number of generators and by Nakayama's lemma that $\mathrm{H}^m(M^\bullet) \otimes_R \mathrm{H}^n(N^\bullet)$ is also nonzero, hence is isomorphic to R. The ring R is local so up to isomorphism R is the only invertible R-module, hence $\mathrm{H}^m(M^\bullet) \cong R$ and $\mathrm{H}^n(N^\bullet) \cong R$. Using the spectral sequence it is now easy to see that $\mathrm{H}^i(M^\bullet) = 0$ whenever $i < m$. The cohomology of $M^\bullet$ is concentrated in degree m where it is isomorphic to R, hence $M^\bullet \cong R[m]$ in $\mathcal{D}_R$. □

There is a canonical Künneth homomorphism of degree 0 [3, IV.6.1]

$$\alpha : \mathrm{H}^*(F^\bullet) \otimes \mathrm{H}^*(G^\bullet) \to \mathrm{H}^*(F^\bullet \otimes G^\bullet).$$

The Künneth homomorphism is preserved by exact functors which commute up to isomorphism with the tensor product. Hence the localization of α at a prime ideal of R is still the Künneth homomorphism.

Lemma 3.2.3. *Let R be a commutative unital ring. If $M^\bullet$ is an invertible object in $\mathcal{D}_R$ then*

$$\oplus_{n\in\mathbb{Z}} H^n(M^\bullet)$$

is an invertible R-module.

Proof. Let $M^\bullet \otimes^{\mathrm{L}}_R N^\bullet \cong R[0]$ in $\mathcal{D}_R$. Localizing the Künneth homomorphism

$$\alpha : \mathrm{H}^*(M^\bullet) \otimes_R \mathrm{H}^*(N^\bullet) \to \mathrm{H}^*(M^\bullet \otimes^{\mathrm{L}}_R N^\bullet)$$

at a prime ideal $\mathfrak{p}$ in R gives us the homomorphism

$$\alpha_{\mathfrak{p}} : \mathrm{H}^*(M^\bullet_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \mathrm{H}^*(N_{\mathfrak{p}}{}^\bullet) \to \mathrm{H}^*(M_{\mathfrak{p}}{}^\bullet \otimes^{\mathrm{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}{}^\bullet).$$

By Proposition 3.2.2 the Künneth homomorphism $\alpha_{\mathfrak{p}}$ is an isomorphism for all prime ideals $\mathfrak{p}$ in R, hence α is an isomorphism. This means that

$$\oplus_{i\in\mathbb{Z}} (\mathrm{H}^i(M^\bullet) \otimes_R \mathrm{H}^{-i}(N^\bullet)) \cong R$$

and

$$\oplus_{i+j=q} (\mathrm{H}^i(M^\bullet) \otimes_R \mathrm{H}^j(N^\bullet)) = 0$$

for all integers $q \neq 0$. Hence we get that

$$(\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(M^\bullet)) \otimes_R (\oplus_{j\in\mathbb{Z}} \mathrm{H}^j(N^\bullet)) \cong R$$

and $\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(M^\bullet)$ is an invertible R-module. □

Lemma 3.2.3 implies that for each prime ideal $\mathfrak{p}$ in R, there is an integer $\Psi(M^\bullet)(\mathfrak{p}) = n$ such that $\mathrm{H}^n(M^\bullet)_\mathfrak{p} \cong R_\mathfrak{p}$ and $\mathrm{H}^q(M^\bullet)_\mathfrak{p} = 0$ for all $q \neq n$. For a topological space T let $C(T)$ denote the additive group of continuous maps from T to the integers $\mathbb{Z}$ with the discrete topology.

We recall a lemma about idempotents.

Lemma 3.2.4. *There is a bijection between the open closed subsets of spec R and the idempotents in R. Let e_U denote the idempotent corresponding to the open closed set U. The bijection has the following properties.*

1. *The subset spec $e_U R \subset$ spec R is U. The open closed set U is $\{\mathfrak{p} \in \operatorname{spec} R \mid e_U R_\mathfrak{p} = R_\mathfrak{p}\}$. If $\mathfrak{q} \notin U$ then $e_U R_\mathfrak{q} = 0$.*

2. *The correspondence gives a natural isomorphism between the Boolean algebra of idempotents in R and the Boolean algebra of open closed subsets in spec R.*

The second statement means that for any open closed sets U_1 and U_2 we have $e_{U_1\cap U_2} = e_{U_1}e_{U_2}$, $e_{U_1\cup U_2 - U_1\cap U_2} = e_{U_1} + e_{U_2} - 2e_{U_1\cap U_2}$, $e_X = 1$ and $e_\emptyset = 0$; naturality here means that if $f : R \to S$ is a map of rings and U is an open closed set in spec R then we have $f(e^R_U) = e^S_{\operatorname{spec}(f)^{-1}(U)}$ in S.

Proof. Let U_1 be an open closed subset of spec R, and let U_2 be the complement of U_1 in spec R. Let I_1 and I_2 be two ideals in R such that $\{\mathfrak{p} \in \operatorname{spec} R \mid \mathfrak{p} \supset I_i\} = U_i$. Then $I_1 \cap I_2$ is in the radical $\sqrt{R}$ of R and $I_1 + I_2 = R$. Let $i_1 \in I_1$ and $i_2 \in I_2$ be elements such that $i_1 + i_2 = 1$. Then their product $i_1 i_2 \in \sqrt{R}$ so there is an n such that $i_1^n i_2^n = 0$. Since $(i_1^n, i_2^n) = R$ there exist elements a and b such that

$a(i_1)^n + b(i_2)^n = 1$ in R. It now follows that $e_1 = a(i_1)^n \in I_2$ and $e_2 = b(i_2)^n \in I_1$ are orthogonal idempotents. The map of spectra induced from $R \to e_i R$ gives an inclusion of U_i into $\operatorname{spec} R$.

We now show that the idempotent e associated to an open closed set U is unique. Let e and e' be two idempotents corresponding to the same open closed set U i.e. $\operatorname{spec}(eR) = \operatorname{spec}(e'R) = U$. The product $e(1-e')$ is in the radical of R hence (e and e' are idempotent) $e(1-e') = 0$; similarly $(1-e)e' = 0$. We now get that

$$e = e(e' + (1-e')) = ee' = (e + (1-e))e' = e'.$$

The other statements in the Lemma follow easily using uniqueness of the idempotent.

□

Theorem 3.2.5. *Let R be a commutative unital ring. There is a natural split short exact sequence*

$$0 \to Pic(R) \to Pic(\mathcal{D}_R) \xrightarrow{\Psi} C(spec\, R) \to 0.$$

Proof. From [15, 4.10] the set of prime ideals such that $\mathrm{H}^n(M^\bullet)_\mathfrak{p} \cong R_\mathfrak{p}$ is an open closed set in $\operatorname{spec} R$. Hence $\mathfrak{p} \mapsto \Psi(M^\bullet)(\mathfrak{p})$ defines a map $\Psi : \operatorname{Pic}(\mathcal{D}_R) \to C(\operatorname{spec} R)$. It is clear from the proof of Proposition 3.2.2 that Ψ is a homomorphism. If $\Psi(M^\bullet) \equiv 0$ then $M^\bullet \cong \mathrm{H}^0(M^\bullet)[0]$ in $\mathcal{D}_R$, and since $\mathrm{H}^0(M^\bullet)$ is an invertible R-module $M^\bullet$ is in the image of $\operatorname{Pic}(R) \to \operatorname{Pic}(\mathcal{D}_R)$.

We construct a splitting of Ψ. The spectrum of R is a compact space so the image of any continuous function $f : \operatorname{spec} R \to \mathbb{Z}$ consists of a finite set of integers, say $n_1, \ldots, n_m$. The disjoint subsets $U_i = f^{-1}(n_i)$ of $\operatorname{spec} R$ are both open and closed, hence correspond to an orthogonal basis of idempotents $e_{U_1}, \ldots, e_{U_m}$ in R. Define the invertible complex $\Phi(f)$ to be $\oplus_{i=1}^m e_{U_i} R[n_i]$. By Lemma 3.2.4 the composite $\Psi \circ \Phi(f)$ is equal to f.

We now check that Φ is a homomorphism. Note that if $f \equiv 0$ then $\Phi(f) \cong R[0]$. For two finite open closed partitions $\{U_i\}$ and $\{V_i\}$ of $\operatorname{spec} R$ we have that

$$\oplus_{i=1}^N e_{U_i} R[n_i] \otimes_R \oplus_{j=1}^M e_{V_j} R[m_j] \cong \oplus_{i,j=1}^{N,M} e_{U_i \cap V_j} R[n_i + m_j].$$

Hence Φ is a homomorphism. It is easy to see that the split short exact sequence is natural. $\square$

3.3 The Picard group of the derived category of $\mathcal{O}$-modules

Recall that $(\mathcal{E}, \mathcal{O})$ is a ringed topos with enough points, and that $\mathcal{M}$ is the category of left $\mathcal{O}$-modules.

Proposition 3.3.1. *Let $F^\bullet$ be an invertible object in the category of left $\mathcal{O}$-modules. Then*

$$\oplus_{n \in \mathbb{Z}} H^n(F^\bullet)$$

is an invertible $\mathcal{O}$-module.

Proof. Let $F^\bullet \otimes^{\mathrm{L}}_{\mathcal{O}} G^\bullet \cong \mathcal{O}$ in $\mathcal{D}_{\mathcal{M}}$. Assume that $F^\bullet$ is a K-flat complex of $\mathcal{O}$-modules as constructed in Lemma 3.1.2. The localization of $F^\bullet$ at a point p of $\mathcal{E}$ is then a K-flat complex of $\mathcal{O}_p$-modules. Taking the stalk at a point p of the Künneth homomorphism α gives

$$\alpha_p : \mathrm{H}^*(F_p^\bullet) \otimes_{\mathcal{O}_p} \mathrm{H}^*(G_p^\bullet) \to \mathrm{H}^*(F_p^\bullet \otimes^{\mathrm{L}}_{\mathcal{O}_p} G_p^\bullet)$$

in the category of $\mathcal{O}_p$-modules. By Lemma 3.2.3 α_p is an isomorphism for all points p. Since $\mathcal{E}$ has enough points the Künneth homomorphism α is an isomorphism. Hence we get as in the proof of Lemma 3.2.3 that

$$(\oplus_{i \in \mathbb{Z}} \mathrm{H}^i(F^\bullet)) \otimes_{\mathcal{O}} (\oplus_{j \in \mathbb{Z}} \mathrm{H}^j(G^\bullet)) \cong \mathcal{O}$$

and $\oplus_{i \in \mathbb{Z}} \mathrm{H}^i(F^\bullet)$ is an invertible $\mathcal{O}$-module. $\square$

Theorem 3.3.2. *Let $(\mathcal{E}, \mathcal{O})$ be a commutative unital ringed Grothendieck topos with enough points such that for all points p of $\mathcal{E}$ the ring $\mathcal{O}_p$ has a connected prime ideal spectrum. Then there is a natural split short exact sequence*

$$0 \to Pic(\mathcal{M}) \to Pic(\mathcal{D}_{\mathcal{M}}) \xrightarrow{\Psi} C(pt(\mathcal{E})) \to 0.$$

Proof. Let $F^\bullet$ be an invertible complex in $\mathcal{D}_{\mathcal{M}}$. By Proposition 3.3.1

$$F = \oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet)$$

is an invertible $\mathcal{O}$-module. Let G be an inverse of F under the tensor product. Let $A^i = \mathrm{H}^i(F^\bullet) \otimes_{\mathcal{O}} G$. Then $\oplus_{i\in\mathbb{Z}} A^i \cong \mathcal{O}$. If we localize this at a point p of $\mathcal{E}$ we get that $\oplus_{i\in\mathbb{Z}} A^i_p \cong \mathcal{O}_p$. From our assumptions on $\mathcal{O}_p$ there is an integer n_p such that $A^{n_p}_p \cong \mathcal{O}_p$ and $A^i_p = 0$ for $i \neq n_p$. Define $\Psi(F^\bullet)(p)$ to be n_p.

If $\Psi(F^\bullet) \equiv 0$ then $\mathrm{H}^0(F^\bullet) \otimes_{\mathcal{O}} G \cong \mathcal{O}$ and $\mathrm{H}^i(F^\bullet) = 0$ for all $i \neq 0$ so $F^\bullet \cong \mathrm{H}^0(F^\bullet)[0] = F[0]$ in $\mathcal{D}_{\mathcal{M}}$.

It remains to prove that Ψ takes values in $C(\mathrm{pt}(\mathcal{E}))$ and is split. We need to prove that if $A \oplus B \cong \mathcal{O}$ then $\{p \in \mathrm{pt}(\mathcal{E}) \mid A_p \cong \mathcal{O}_p\}$ is an open closed set in $\mathrm{pt}(\mathcal{E})$. Denote the terminal object in $\mathcal{E}$ by $\bullet$. The sheaf of sets $\bullet$ associates to every object in the site $\mathcal{C}$ the one-point set.

Let $1 : \bullet \to \mathcal{O}$ be the unit and $0 : \bullet \to \mathcal{O}$ the zero element. We can compose these two elements with the projection from $\mathcal{O} \cong A \oplus B$ to A.

There is an equalizer

$$S \longrightarrow \bullet \underset{0}{\overset{1}{\rightrightarrows}} A$$

in $\mathcal{E}$, and S is a subobject of $\bullet$. Points preserve limits so we get for each point p of $\mathcal{E}$ an equalizer

$$S_p \longrightarrow \bullet \underset{0}{\overset{1}{\rightrightarrows}} A_p$$

in the category of sets. Since $S_p \neq \emptyset$ if and only if $1 = 0$ in A_p it follows that

$$\{p \in \mathrm{pt}(\mathcal{E}) \mid A_p = 0\} = \{p \in \mathrm{pt}(\mathcal{E}) \mid S_p = \bullet\}$$

which by definition is an open set in $\mathrm{pt}(\mathcal{E})$ [10, 7.8]. The same argument applied to B shows that $\{p \in \mathrm{pt}(\mathcal{E}) \mid B_p = 0\}$ is also open in $\mathrm{pt}(\mathcal{E})$. Since $A_p \oplus B_p \cong \mathcal{O}_p$ and $\mathcal{O}_p$ has connected prime ideal spectrum exactly one of A_p and B_p is zero, so the two sets $\{p \in \mathrm{pt}(\mathcal{E}) \mid A_p = 0\}$ and $\{p \in \mathrm{pt}(\mathcal{E}) \mid B_p = 0\}$ are complements of each other in $\mathrm{pt}(\mathcal{E})$. Hence $\{p \in \mathrm{pt}(\mathcal{E}) \mid A_p = 0\}$ is an open closed set in $\mathrm{pt}(\mathcal{E})$.

We now construct a splitting of Ψ. Let ϕ be a continuous function $\mathrm{pt}(\mathcal{E}) \to \mathbb{Z}$. For each integer n let S_n be the subobject of $\bullet$ corresponding to the open closed set $\phi^{-1}(n)$. For a subobject S in $\bullet$, let $\mathcal{O}_S$ be defined by $\mathcal{O}_S(X) = \mathcal{O}(X)$ if $S(X) = \bullet$ and $\mathcal{O}_S(X) = 0$ if $S(X) = \emptyset$ for $X \in \mathcal{C}$. By considering the zero and the unit maps $0, 1 : S_i \to \mathcal{O}_{S_i}$, using the evident isomorphism $\oplus_{i\in\mathbb{Z}}\mathcal{O}_{S_i} \to \mathcal{O}$ and our assumption on $\mathcal{O}_p$ it is easy to see that $(\mathcal{O}_{S_i})_p \cong \mathcal{O}_p$ if and only if $(S_i)_p = \bullet$. Define the complex $\Phi(\phi)$ to be $\mathcal{O}_{S_i}$ in degree i and to have trivial differentials. Since $\mathcal{O}_S \otimes_{\mathcal{O}} \mathcal{O}_T = \mathcal{O}_{S\cap T}$ for two subobjects S and T of $\bullet$, it follows that Φ is a homomorphism. In particular

$$\Phi(\phi) \otimes_{\mathcal{O}} \Phi(-\phi) \cong \Phi(0) = \oplus_{i\in\mathbb{Z}}\mathcal{O}_{S_i} \cong \mathcal{O}.$$

So Φ takes values in $\mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$. Since $(\mathcal{O}_{S_i})_p \cong \mathcal{O}_p$ if and only if $(S_i)_p = \bullet$, the composite $\Psi \circ \Phi$ is the identity on $C(\mathrm{pt}(\mathcal{E}))$. It is easy to see that all three maps in the split short exact sequence are natural with respect to maps of ringed topoi. $\square$

As a special case we get the following result.

Corollary 3.3.3. *Let $(X, \mathcal{O})$ be a locally ringed space with an action by a discrete group G [9]. There is a natural split short exact sequence*

$$0 \to Pic_G(X) \to Pic(\mathcal{D}_{sh_G(X)}) \xrightarrow{\Psi} C^G(X) \to 0.$$

Here $\mathrm{sh}_G(X)$ denotes the category of left G-$\mathcal{O}$-modules, $\mathrm{Pic}_G(X)$ denotes the group of isomorphism classes of invertible G-$\mathcal{O}$-modules, and $C^G(X)$ denotes the G-fixed subgroup of $C(X)$ [10, 8.4.1].

We now generalize the theorem to ringed topoi where the $\mathcal{O}_p$ are not necessarily connected. Define a sheaf of abelian groups $C(\mathcal{O})$ by sheafifying the presheaf which maps an object $X \in \mathcal{C}$ to $C(\mathrm{spec}\,\mathcal{O}(X))$. Let $\Gamma(G) = \hom_{\mathcal{E}}(\bullet, G)$ denote the global sections functor.

Proposition 3.3.4. *There is a natural split short exact sequence*

$$0 \to Pic(\mathcal{M}) \to Pic(\mathcal{D}_{\mathcal{M}}) \xrightarrow{\tilde{\Psi}} \Gamma(C(\mathcal{O})) \to 0.$$

Proof. Let $F^\bullet$ be an invertible complex in $\mathcal{D}_\mathcal{M}$. From Proposition 3.3.1 $F = \oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet)$ is an invertible $\mathcal{O}$-module. The direct sum (resp. tensor product) in $\mathcal{M}$ is the sheafification of the presheaf direct sum (resp. tensor product). Let $F \otimes_\mathcal{O} G \cong \mathcal{O}$. There is a covering $\{V_\gamma\}$ such that

$$\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet)(V_\gamma) \otimes_{\mathcal{O}(V_\gamma)} G(V_\gamma) = (\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet) \otimes_\mathcal{O} G))(V_\gamma) \cong \mathcal{O}(V_\gamma)$$

for each γ. Define $\psi_\gamma : \operatorname{spec} \mathcal{O}(V_\gamma) \to \mathbb{Z}$ by letting $\psi_\gamma(\mathfrak{p})$ be the unique integer i such that $\mathrm{H}^i(F^\bullet)(V_\gamma)_\mathfrak{p} \neq 0$. The maps $\{\psi_\gamma\}$ are compatible so they define an element $\tilde{\Psi}(F^\bullet)$ in $\Gamma(C(\mathcal{O}))$. It is easy to see that the map $\tilde{\Psi}$ is a homomorphism.

We now construct a splitting Φ of $\tilde{\Psi}$. Given $\phi \in \Gamma(C(\mathcal{O}))$, there exists a covering $\{V_\gamma\}$ and maps $\phi_\gamma : \operatorname{spec} \mathcal{O}(V_\gamma) \to \mathbb{Z}$ such that $\{\phi_\gamma\} = \phi$ in $\Gamma(C(\mathcal{O}))$. By Lemma 3.2.4 there are unique idempotents $e^n_\gamma \in \mathcal{O}(V_\gamma)$ such that for each e^n_γ the subspace $\operatorname{spec}(e^n_\gamma \mathcal{O}(V_\gamma)) \subset \operatorname{spec} \mathcal{O}(V_\gamma)$ is $\phi_\gamma^{-1}(n)$. For a given n let $e^n = \{e^n_\gamma\}$. Then e^n is an idempotent element in $\Gamma(\mathcal{O})$, $e^n e^m = 0$ for $n \neq m$, and $\oplus_{i\in\mathbb{Z}} e^n \mathcal{O} \to \mathcal{O}$ is an isomorphism of $\mathcal{O}$-modules. Define $\Phi(\phi)$ to be the complex which is $e^n\mathcal{O}$ in degree n and has trivial differentials. Then Φ is a homomorphism from $\Gamma(C(\mathcal{O}))$ to $\mathrm{Pic}(\mathcal{D}_\mathcal{M})$ such that $\tilde{\Psi} \circ \Phi$ is the identity on $\Gamma(C(\mathcal{O}))$. The naturality is easily verified. $\square$

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