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① 16.4 Delbrøkoppspalting (via eksempler)
(antideriverte til rasjonale funksjoner)

Alle polynomer av grad n har antideriverte som er polynomer av grad $n+1$.

$$\left(\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1 \right)$$

Antideriverte til rasjonale funksjoner på formen

$\frac{p(x)}{ax+b}$. $\qquad p(x)$ polynom :

Ved polynomdivisjon : $\frac{p(x)}{ax+b} = S(x) + \frac{k}{ax+b}$

$S(x)$ polynom ..

$\int \frac{1}{ax+b} \, dx$ $\qquad$ bruker lineær substitusjon

$U = ax+b$

$U' = a \qquad du = a\,dx$

$\frac{1}{a} du = dx$

$$\int \frac{1}{U} \frac{1}{a} \, dU$$

$$= \frac{1}{a} \ln|U| + c = \frac{1}{a} \ln|ax+b| + c.$$

$$\int \frac{p(x)}{ax+b} \, dx = \int S(x) + k \cdot \frac{1}{ax+b} \, dx$$

$$= \underline{\int S(x)\,dx + k \cdot \frac{1}{a} \ln|ax+b| + c}.$$

②

ex $\int \frac{x^2+1}{x-1}\,dx$

polynomdivisjon :

$$\begin{array}{l} x^2+1 : x-1 = x+1 \\ \underline{(x^2-x)} \\ \quad x+1 \\ \quad \underline{x-1} \\ \qquad 2 \end{array}$$

rest

$$\int \frac{x^2+1}{x-1}\,dx = \int x+1 + \frac{2}{x-1}\,dx$$

$$= \frac{x^2}{2} + x + 2\int \frac{1}{x-1}\,dx$$

$$= \underline{\frac{x^2}{2} + x + 2\ln|x-1| + c}$$

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ex $\int \frac{2x^3+x^2-8x}{2x-3}\,dx$

polynomdivisjon :

$$\begin{array}{l} 2x^3+x^2-8x : 2x-3 = x^2+2x-1 \\ \underline{2x^3-3x^2} \\ \quad 4x^2-8x \\ \quad \underline{4x^2-6x} \\ \qquad -2x \\ \qquad \underline{-2x+3} \\ \qquad\quad -3 \end{array}$$

$$\int \frac{2x^3+x^2-8x}{2x-3}\,dx$$

$$= \int x^2+2x-1 - \frac{3}{2x-3}\,dx$$

$$= \underline{\frac{x^3}{3} + x^2 - x - \frac{3}{2}\ln|2x-3| + c}$$

oppg $\int \frac{x^2}{2x+1}\,dx$

(3) $= \int \frac{1}{2}x - \frac{1}{4} + \frac{1/4}{2x+1}\,dx$

$= \frac{x^2}{4} - \frac{x}{4} + \frac{1}{4}\cdot\frac{1}{2}\ln|2x+1| + C$

$= \frac{1}{4}\left(x^2 - x\right) + \frac{1}{8}\ln|2x+1| + C$

Eksempler på delbrøksoppspalting:

$\frac{1}{x(x-1)}$ er lik $\frac{A}{x} + \frac{B}{x-1}$

for konstanter A og B.

Vi bestemmer A og B:

$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$ (finner fellesnevner)

$= \frac{A(x-1) + Bx}{x(x-1)}$, sammenligner tellerne

$1 = A(x-1) + Bx$. (for alle x)

Metode 1: "setter inn gunstige x-verdier og løser for A og B"

setter $x = 1$: $1 = A\cdot 0 + B\cdot 1 = B$

$x = 0$: $1 = A(-1) + B\cdot 0 = -A$

så $B = 1$ og $A = -1$.

Metode 2: $0\cdot x + 1 = (A+B)x - A$

så $A + B = 0$ og $1 = -A$.

Derfor er $A = -1$ og $B = 1$.

sammenligner koeffisientene til polynomene i tellerne.

④ $$\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$$

$$\int \frac{1}{x(x-1)}\,dx = \int \frac{1}{x-1} - \frac{1}{x}\,dx$$

$$= \ln|x-1| - \ln|x| + c$$

$$= \underline{\ln\left|\frac{x-1}{x}\right| + c}$$

ex $$\int \frac{1}{x^2-1}\,dx$$

$$= \int \frac{1}{(x+1)(x-1)}\,dx$$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$ delbrøksoppspalting

$$= \frac{A(x-1) + B(x+1)}{(x+1)(x-1)}$$

Derfor er $1 = A(x-1) + B(x+1)$

Setter $x=1$: $1 = A\cdot 0 + B\cdot 2$ så $B = \frac{1}{2}$

$x=-1$: $1 = -2A + 0$ så $A = \frac{-1}{2}$

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{2}\left(\frac{-1}{x+1} + \frac{1}{x-1}\right)dx$$

$$= \frac{1}{2}\left(-\ln|x+1| + \ln|x-1|\right) + c$$

$$= \underline{\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c}$$

⑤

ex $\displaystyle\int \frac{-x+2}{2x^2+x-1}\,dx$

Faktorisere $2x^2+x-1$: -1 er en rot. så $(x+1)$ er en faktor.

$$2x^2+x-1 = (x+1)(2x-1)$$

Delbrøksoppspalting: $\displaystyle\frac{-x+2}{(2x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{2x-1}$

$$= \frac{A(2x-1)}{(2x-1)(x+1)} + \frac{B(x+1)}{(2x-1)(x+1)}$$

$$-x+2 = A(2x-1) + B(x+1)$$

Setter inn $x=-1$: $3 = A(-3) + B\cdot 0$ så $\underline{A=-1}$.

$x = \frac{1}{2}$: $-\frac{1}{2}+2 = A\cdot 0 + B\left(\frac{3}{2}\right)$

$\frac{3}{2} = B\left(\frac{3}{2}\right)$ så $\underline{B=1}$

(alternativt: $-x+2 = (2A+B)\cdot x + (-A+B)$)

$$\int \frac{-x+2}{2x^2+x-1}\,dx = \int \frac{-1}{x+1} + \frac{1}{2x-1}\,dx$$

$$= -\ln|x+1| + \frac{1}{2}\ln|2x-1| + c$$

$$= \underline{\ln\left(\frac{\sqrt{|2x-1|}}{|x+1|}\right) + c}$$

oppg $\int \frac{5x+4}{x^2+x-2}\,dx$

(6)

$x=1$ er en rot til x^2+x-2, så $(x-1)$ er en faktor i x^2+x-2.

$$x^2+x-2 = (x+2)(x-1)$$

$$\frac{5x+4}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$5x+4 = A(x-1) + B(x+2)$$

Setter inn $x=1$: $9 = A\cdot 0 + B\cdot 3$ så $\underline{B=3}$

$x=-2$: $-6 = A(-3)$ så $\underline{A=2}$

$$\int \frac{5x+4}{x^2+x-2}\,dx = \int \frac{2}{x+2} + \frac{3}{x-1}\,dx$$

$$= \underline{2\ln|x+2| + 3\ln|x-1| + c}$$

(7) Hvis graden til teller i et rasjonalt uttrykk er større eller lik graden til nevneren utføres polynom divisjon først.

$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

p, q og s polynomer $\qquad \deg r < \deg q.$

(graden)

ex $\displaystyle\int \frac{x^3+2}{x^2-1}\,dx$

polynomdivisjon : $\quad x^3+2 : x^2-1 = x$

$$\begin{array}{l} x^3 - x \\ \hline x+2 \end{array}$$

$$\frac{x^3+2}{x^2-1} = x + \frac{x+2}{x^2-1}$$

Delbrøks-oppspalting: $\displaystyle\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$

tellerne : $\quad x+2 = A(x+1) + B(x-1)$

Setter inn $x=1 \qquad 3 = A\cdot 2 \qquad$ så $\quad A = \frac{3}{2}$

$x=-1 \qquad 1 = B(-2) \qquad$ så $\quad B = \frac{-1}{2}$

$$\int \frac{x^3+2}{x^2-1}\,dx = \int x + \frac{3}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}\,dx$$

$$= \frac{x^2}{2} + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$$

$$= \underline{\frac{1}{2}\left(x^2 + 3\ln|x-1| - \ln|x+1|\right) + c}$$

Substitusjon kan i blant forenkle integralene

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$$\int \frac{1}{x(x^2-1)}\,dx = \int \frac{1}{x(x-1)(x+1)}\,dx$$

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$$

Løsningen er $A = -1, \quad B = C = \frac{1}{2}$.

$$\int \frac{1}{x(x^2-1)}\,dx = \int \frac{-1}{x} + \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x+1}\right) dx$$

$$= -\ln|x| + \frac{1}{2}\ln|(x-1)(x+1)| + c$$

$$= \frac{1}{2}\left(\ln|x^2-1| - 2\ln|x|\right) + c = \underline{\frac{1}{2}\ln\left|\frac{x^2-1}{x^2}\right| + c}$$

Alternativt kan vi bruke substitusjon:

La $u = x^2$ $\qquad \frac{du}{dx} = 2x$

$$du = 2x\,dx$$

$$\int \frac{1}{x(x^2-1)}\,dx = \int \frac{2x}{2x^2(x^2-1)}\,dx$$

(substitusjon)

$$= \int \frac{du}{2u(u-1)} = \int \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

$$= \frac{1}{2}\left(\ln|u-1| - \ln|u|\right) + c$$

$$= \underline{\frac{1}{2}\ln\left|\frac{x^2-1}{x^2}\right| + c}$$

(Generelt: $\displaystyle\int \frac{1}{x} r(u)\,dx = \int \frac{nx^{n-1}}{nx^n} r(u)\,dx$

$\displaystyle = \frac{1}{n}\int \frac{1}{u}\cdot r(u)\,du$ hvor $u = x^n$.)