11 april 2013

Delvis integrasjon

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Produktregelen

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Så $u \cdot v$ er en antiderivert til $u' \cdot v + u \cdot v'$

$$\int u' \cdot v + u \cdot v' \, dx = u \cdot v + C$$

$$= \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

$$\boxed{\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx}$$

Gjør om et integral til et annet forhåpentligvis enklere integral.

Eksempler

$$\int x \sin x \, dx$$

$x = u$, $\sin x = v'$ velger $v = -\cos x$

Bruker delvis integrasjon

$$\int \overset{u}{x} \, \overbrace{\sin x}^{v'} \, dx = x(-\cos x) - \int \overset{u'}{1} (-\cos x) \, dx$$

$$= -x \cdot \cos(x) + \int \cos x \, dx$$

$$= \underline{-x \cos x + \sin x + C}$$

(test: $(-x \cos x)' + (\sin x)'$

$= (-x)(-\sin x) + (-1) \cos x + \cos x$

$= x \sin(x) + 0 = \underline{x \sin x}$ ✓)

② $\int x \sin x \, dx$

Velger en annen måte å anvende delvis integrasjon

$x = v' \qquad \sin x = u$

Velger $v = \frac{x^2}{2} \qquad u' = \cos x$

$$\int \overset{v'}{x} \overset{u}{\sin x} \, dx = \overset{v}{\frac{x^2}{2}} \overset{u}{\sin x} - \int \overset{v}{\frac{x^2}{2}} \overset{u'}{\cos x} \, dx$$

Hva er $\int x^2 \cos x \, dx$?

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx$$

$$= x^2 \sin x - 2(-x \cos x + \sin x) + c$$

$$= \underline{x^2 \sin x + 2x \cos x - 2 \sin x + c}$$

Delvis integrasjon to ganger.

oppgave Finn $\int x e^{-x} \, dx$

$\int x^2 e^{-x} \, dx$.

$\int x e^{-x} \, dx \qquad u = x \qquad u' = 1$

$v' = e^{-x} \qquad v = -e^{-x}$

$$\int \overset{u}{x} \overset{v'}{e^{-x}} \, dx = \overset{u}{x} \overset{v}{(-e^{-x})} - \int \overset{u'}{1} \overset{v}{(-e^{-x})} \, dx$$

$$= -x e^{-x} + \int e^{-x} \, dx$$

$$= -x e^{-x} - e^{-x} + c$$

$$= \underline{-e^{-x}(x+1) + c}$$

(3) $\int x^2 e^{-x}\,dx$ $\quad u = x^2 \quad u' = 2x$
$\quad v' = e^{-x} \quad v = -e^{-x}$

$$\int x^2 e^{-x}\,dx = x^2\cdot(-e^{-x}) - \int 2x\cdot(-e^{-x})\,dx$$
$$= -x^2 e^{-x} + 2\int x e^{-x}\,dx$$
$$= \left[-x^2 - 2(x+1)\right] e^{-x} + c$$
$$= \underline{-(x^2+2x+2)\,e^{-x} + c}$$

$\int x^n e^{-x}\,dx$ $\quad u = x^n \quad u' = n x^{n-1}$
$\quad v' = e^{-x} \quad v = -e^{-x}$

$$\int x^n e^{-x}\,dx = x^n(-e^{-x}) - \int n\cdot x^{n-1}(-e^{-x})\,dx$$

$$\boxed{\underline{\int x^n e^{-x}\,dx} = -x^n e^{-x} + n\underline{\int x^{n-1} e^{-x}\,dx}}$$

Rekursive formler

$n=5$

$$\int x^5 e^{-x}\,dx = -x^5 e^{-x} + 5\int x^4 e^{-x}\,dx$$
$$= -x^5 e^{-x} + 5\left[-x^4 e^{-x} + 4\int x^3 e^{-x}\,dx\right]$$
$$= -x^5 e^{-x} - 5x^4 e^{-x} + 5\cdot4\left[-x^3 e^{-x} + 3\int x^2 e^{-x}\,dx\right]$$
$$= -x^5 e^{-x} - 5x^4 e^{-x} - 5\cdot4\,x^3 e^{-x} + 5\cdot4\cdot3\left[-x^2 e^{-x} + 2\int x e^{-x}\,dx\right]$$
$$= -e^{-x}(x^5 + 5x^4 + 5\cdot4x^3 + 5\cdot4\cdot3\cdot x^2) + 5\cdot4\cdot3\cdot2\left[-x e^{-x} + \underbrace{\int e^{-x}\,dx}_{-e^{-x}}\right]$$
$$= \underline{-e^{-x}\left[x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120\right]}$$

(4)

$$\int \overbrace{(3x+1)}^{u} \overbrace{e^{2x-3}}^{v'} dx \qquad u = 3x+1 \qquad u' = 3$$
$$v' = e^{2x-3} \qquad v = \frac{1}{2}e^{2x-3}$$

$$= \overbrace{(3x+1)}^{u} \overbrace{\frac{1}{2}e^{2x-3}}^{v} - \int \overbrace{3}^{u'} \cdot \frac{1}{2}e^{2x-3}\, dx$$

$$= \frac{3x+1}{2} e^{2x-3} - \frac{3}{2} \cdot \frac{e^{2x-3}}{2} + c$$

$$= \left(\frac{3x}{2} + \frac{1}{2} - \frac{3}{4}\right) e^{2x-3} + c$$

$$= \underline{\left(\frac{3x}{2} - \frac{1}{4}\right) e^{2x-3} + c}.$$

* $\int x^3 e^{-x^2} dx$

Substitusjon : $u = x^2$

$du = 2x\,dx \,, \quad \frac{1}{2}du = x\,dx$

$$\int x^3 e^{-x^2} dx = \int x^2 e^{-x^2} \cdot x\,dx$$

$$= \int u e^{-u} \cdot \frac{1}{2} du$$ Delvis integrasjon

$$= \frac{1}{2}\left(-u e^{-u} - e^{-u}\right) + c$$ $u = x^2$

$$= \underline{-\frac{1}{2} e^{-x^2}(x^2+1) + c}$$

(5) Vi løser integralet

$\int \sin x \cdot \cos x \, dx$ på ulike måter:

Substitusjon 1) $u = \sin x$ $\quad u' = \cos x$

$$\int u \cdot u' \, dx = \int u \, du = \frac{1}{2}u^2 + c$$
$$= \underline{\frac{1}{2}\sin^2 x + c}$$

2) $V = \cos x$ $\quad V' = -\sin x$

$$\int V \cdot (-V') \, dx = -\int V \, dV = -\frac{V^2}{2} + c$$
$$= \underline{\frac{-1}{2}\cos^2 x + c}$$

3) Trigonometrisk identitet: $\sin x \cdot \cos x = \frac{1}{2}\sin(2x)$

$$\int \sin x \cos x \, dx = \int \frac{1}{2}\sin(2x) \, dx = \frac{1}{2}\int \sin(2x) \, dx$$
$$= \frac{1}{2}\left(\frac{-1}{2}\cos(2x)\right) = \underline{\frac{-1}{4}\cos(2x)} + c$$

Svarene er like:

$$\sin^2 x = 1 - \cos^2 x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$
$$-\frac{1}{4}\cos(2x) = \frac{-1}{4}(2\cos^2 x - 1) = \frac{-1}{2}\cos^2 x + \frac{1}{4}$$

4) Delvis integrasjon

$$\int \overset{u}{\sin x} \, \overset{v'}{\cos x} \, dx \qquad u' = \cos x, \quad v = \sin x$$
$$= \overset{u}{\sin x} \cdot \overset{v}{\sin x} - \int \overset{u'}{\cos x} \cdot \overset{v}{\sin x} \, dx$$

Så $\quad 2\int \sin x \cdot \cos x \, dx = \sin^2 x + c$

$$\int \sin x \cdot \cos x \, dx = \frac{1}{2}\sin^2 x + c$$

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$$\int \ln|x|\,dx = \int \overset{v'}{1} \cdot \overbrace{\ln|x|}^{u}\,dx$$

$u = \ln|x| \qquad u' = 1/x$

$v' = 1 \qquad v = x$

Delvis integrasjon:

$$\int \ln|x|\,dx = \overset{v}{x} \cdot \overset{u}{\ln|x|} - \int \underbrace{\overset{v}{x} \cdot \overset{u'}{\frac{1}{x}}}_{1}\,dx$$

$$\int \ln|x|\,dx = \underline{x \ln|x| - x + C}$$

$$\int x^4 \ln|x|\,dx = \frac{x^5}{5}\left(\ln|x| - \frac{1}{5}\right) + C$$

Prøver med $\quad u = \ln|x| \qquad u' = 1/x$

$\qquad\qquad\quad v' = x^4 \qquad v = x^5/5$

$$\int \overset{v'}{x^4} \overset{u}{\ln|x|}\,dx = \frac{x^5}{5} \cdot \ln|x| - \int \overset{v}{\frac{x^5}{5}} \cdot \overset{u'}{\frac{1}{x}}\,dx$$

$$= \frac{x^5}{5} \ln|x| - \frac{1}{5}\int x^4\,dx$$

$$= \underline{\frac{x^5}{5} \ln|x| - \frac{1}{5} \cdot \frac{x^5}{5} + C}$$

(7)

$$\int \overset{u}{e^x} \overset{v'}{\cos x}\, dx \qquad u = e^x \quad u' = e^x \qquad v' = \cos x \quad v = \sin x$$

$$= \overset{u}{e^x} \cdot \overset{v}{\sin x} - \int \underset{w}{\overset{u'}{e^x}} \underset{z'}{\overset{v}{\sin x}}\, dx$$

$$\int \overset{w}{e^x} \overset{z'}{\sin x}\, dx = \overset{w}{e^x} \overset{z}{(-\cos x)} - \int e^x(-\cos x)\, dx$$

$$= -e^x \cos x + \int e^x \cos x\, dx$$

Derfor er

$$\int e^x \cos x\, dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\, dx\right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

Flytter integralet på høyre side over til venstre side :

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + c$$

$$\int e^x \cos x\, dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + c$$

$$= \underline{\frac{e^x}{2}(\sin x + \cos x) + c}$$