11.01.2021

Derivasjon.

$f(x)$ definisjonsmengde D_f

f er <u>deriverbar</u> i $a \in D_f$ hvis

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{eksisterer}$$

Grensen er den deriverte til f i $x = a$

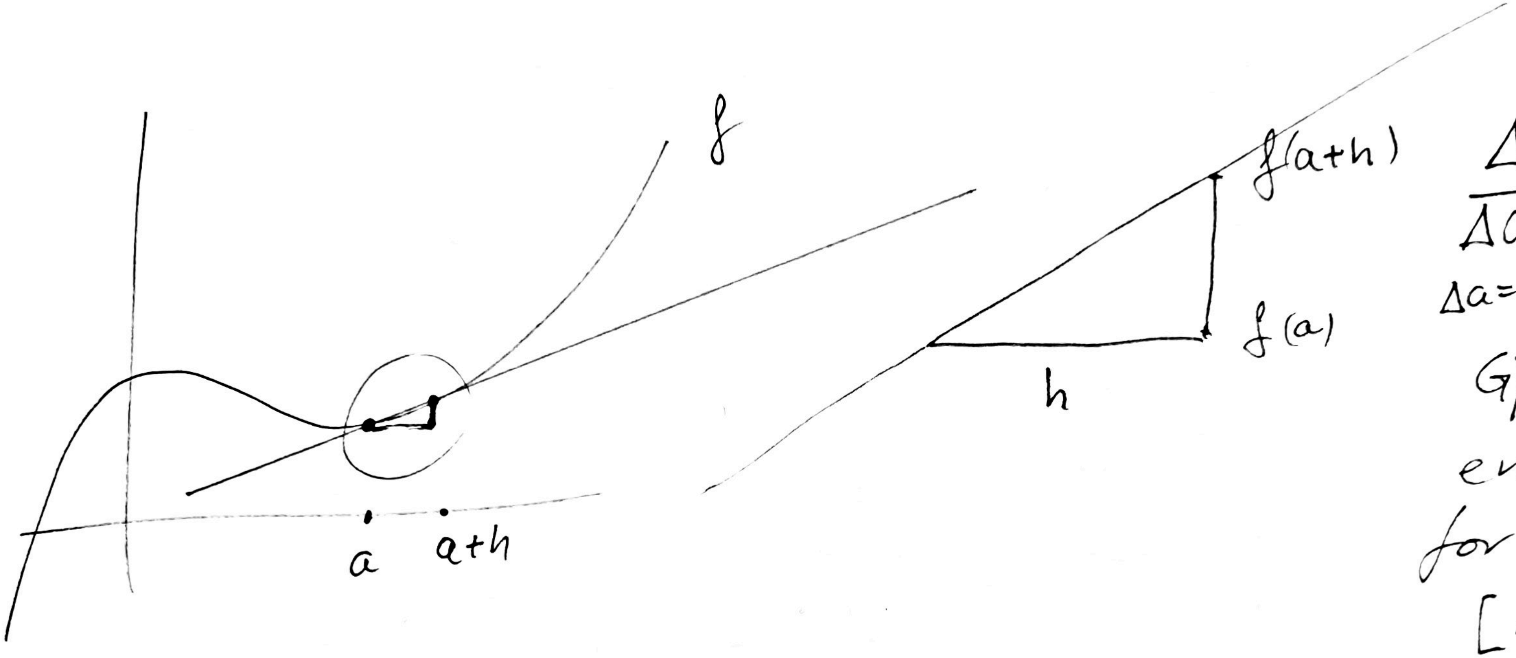
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Ny funksjon: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ $x \in D_f$ hvor f er deriverbar

Alternative skrive-måter

$$f'(x) = \frac{df}{dx}(x) = D_x f = \dot{f}(x)$$

(brukes helst om tids-derivert)



Stigningstallet

$$\frac{\Delta f}{\Delta a} = \frac{f(a+h) - f(a)}{h}$$

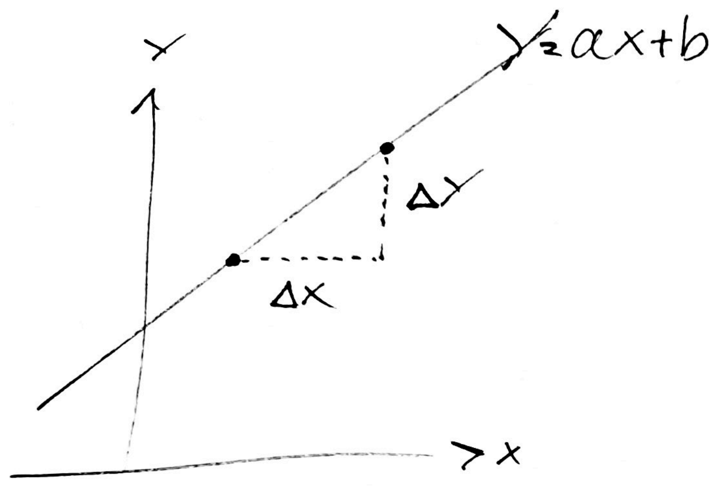
$\Delta a = h$.

Gjennomsnittlig endringsrate for f i intervallet $[a, a+h]$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

momentane endringsraten.

(Endringsrate = vekstfart.)

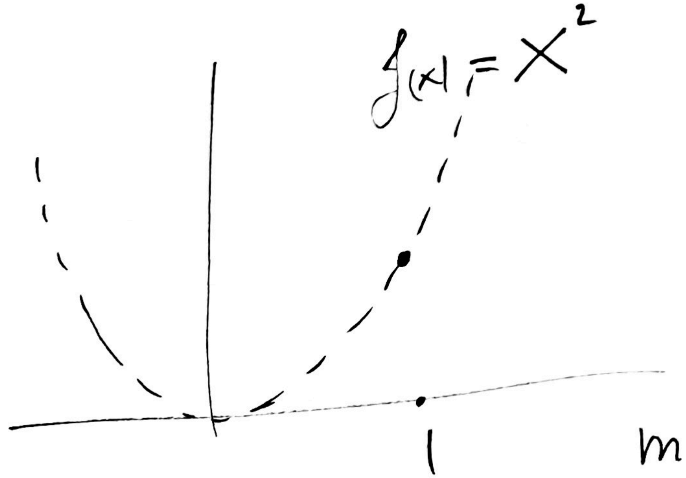


$$\frac{\Delta y}{\Delta x} = a \text{ for alle } x$$

$$\frac{f(x+h)-f(x)}{\Delta x} \qquad (\Delta x = h)$$

$$\lim_{h(=\Delta x)\to 0} \underbrace{\frac{f(x+h)-f(x)}{h}}_{a} = a.$$

$$\boxed{(ax+b)' = a}$$



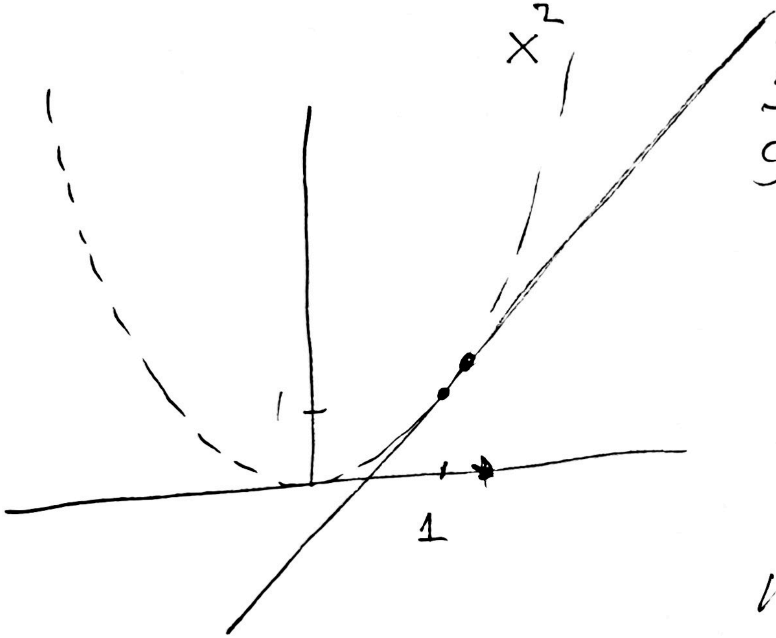
$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h} = \frac{h(2x+h)}{h}$$

$$= 2x + h \quad \text{gjennomsnittlig endringsrate.}$$

momentan endingsrate $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2x+h)$

$$= 2x.$$



<u>Tangentlinje</u> til $f(x)$ i x
går gjennom $(x, f(x))$ og
har stigningstall $f'(x)$.

sekantlinjene gjennom
$(x, f(x))$ og $(x+h, f(x+h))$
nærmer seg tangentlinjen
i $(x, f(x))$ når $h \to 0$.

Resultat $\quad (x^n)' = n\, x^{n-1}$. $\quad$ gyldig for alle reelle n !

n=2 $\quad (x^2)' = 2\, x^{2-1} = 2x^1 = 2x$ ✓

$(x^3)' = 3\, x^{3-1} = 3x^2$

sjekker dette:

$$(x^3)' = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$\begin{aligned}(x+h)^3 &= (x+h)^2 \cdot (x+h) \\ &= (x^2+2xh+h^2)(x+h) \\ &= x^3+2x^2h+h^2x \\ &\quad x^2h+2xh^2+h^3 \end{aligned} = x^3+3x^2h+3xh^2+h^3.$$

$$(x^3)' = \lim_{h \to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h \to 0} (3x^2+3xh+h^2)$$

$$= 3x^2 \checkmark$$

Bevis for $(x^n)' = nx^{n-1}$ for naturlige tall n.

$$(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1-x^n \qquad x = \frac{a}{b}$$

$$\left(1-\frac{a}{b}\right)\left(1+\frac{a}{b}+\left(\frac{a}{b}\right)^2+\cdots+\left(\frac{a}{b}\right)^{n-1}\right) = 1-\left(\frac{a}{b}\right)^n \quad | \cdot b^n$$

$$(b-a)(b^{n-1}+ab^{n-2}+a^2b^{n-3}+\cdots+a^{n-1}) = b^n - a^n$$

La nå $b = x+h$ og $a = x$

$$(x+h)^n - x^n = h\left((x+h)^{n-1} + x(x+h)^{n-2} + x^2(x+h)^{n-3} + \cdots + x^{n-1}\right)$$

$$(x^n)' = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \underbrace{(x+h)^{n-1} + x(x+h)^{n-2} + \cdots + x^{n-1}}_{n \text{ ledd}}$$

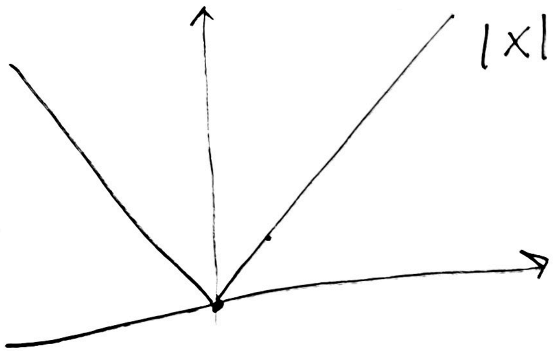
$$\boxed{(x^n)' = nx^{n-1}} \qquad (n \geq 1 \text{ heltall})$$

eks:

$$(x^7)' = 7x^{7-1} = 7x^6 \qquad (x^{231})' = 231x^{230}$$

$$(x^1)' = 1 \cdot x^{1-1} = 1 \quad \checkmark$$

$$x^0 = 1 \text{ konstant} \qquad (x^0)' = 0x^{0-1} = 0 \quad \checkmark$$



$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$|x|$ er <u>ikke</u> deriverbar i 0.

den har et <u>knekkpunkt</u> i $x=0$.

$$\lim_{h\to 0^-} \frac{|x+h|-|x|}{h}\Big|_{x=0} = -1$$

$$\lim_{h\to 0^+} \frac{|x+h|-|x|}{h}\Big|_{x=0} = +1$$

eksisterer men er ulike.

Hvis $f(x)$ er deriverbar i $x=a$, da er $f(x)$ også kontinuerlig i $x=a$.

Deriverba $\Rightarrow$ Kontinuerlig.

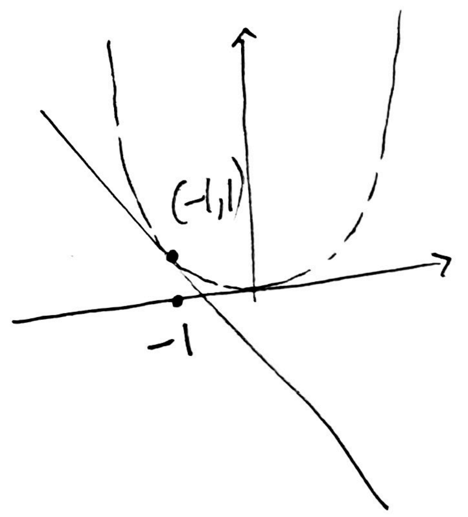
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\cdot h = \lim_{h\to 0} f(x+h)-f(x)$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\cdot \lim_{h\to 0} h = f'(x)\cdot 0 = 0$$

så f er kontinuerlig.

så $\lim_{h\to 0} f(x+h) = f(x)$

Finn tangentlinjen $y = ax+b$ til $f(x) = x^4$ i $x=-1$.



$f(-1) = (-1)^4 = 1$

$f'(x) = 4x^{4-1} = 4x^3$.

$f'(-1) = 4(-1)^3 = -4$

Linjen har $a = -4$ (stigningstall)

og går gjennom $(-1, 1)$.

$y = a(x - x_0) + y_0$

$= -4(x-(-1)) + 1$

$= -4(x+1) + 1 = -4x - 4 + 1$

$y = \underline{-4x - 3}$

Derivasjon er lineær.

$$\left(f(x) + g(x)\right)' = f'(x) + g'(x)$$

$$\left(k f(x)\right)' = k \cdot f'(x)$$

Egenskapene følger fra definisjonen

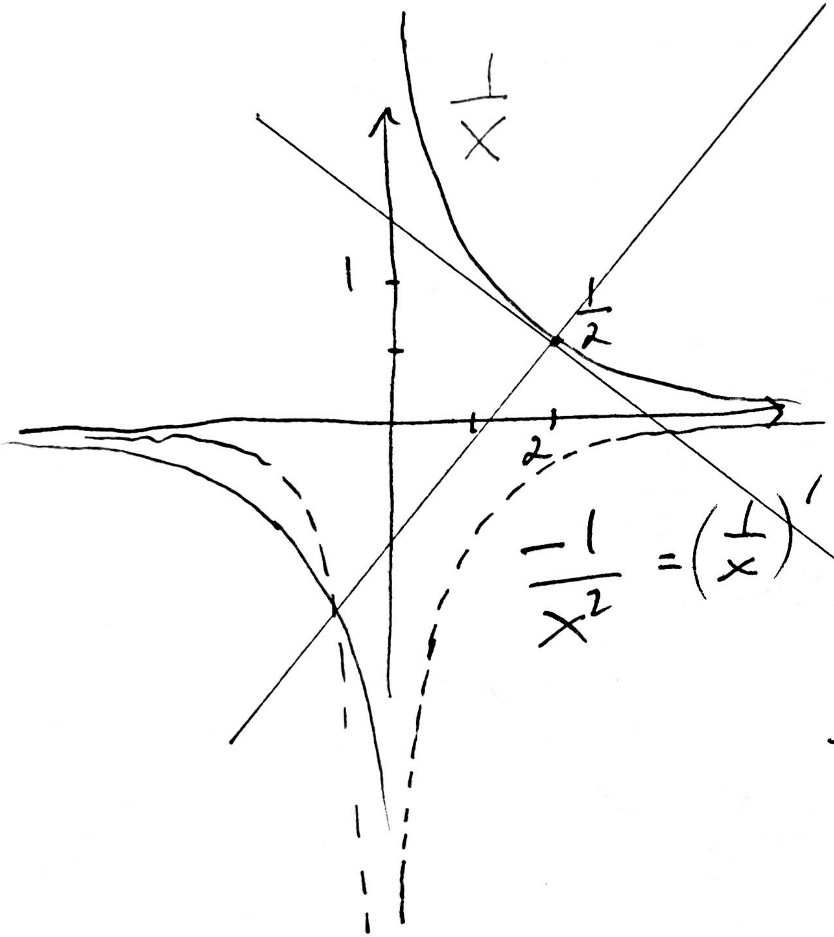
Eks

$$\begin{aligned}(3x - 4x^5 + 2x^7)' &= (3x)' + (-4x^5)' + (2x^7)' \\ &= 3(x)' + (-4)(x^5)' + 2(x^7)' \\ &= 3 \cdot 1 - 4(5x^4) + 2(7x^6) \\ &= \underline{3 - 20x^4 + 14x^6}\end{aligned}$$

oppg. Finn den deriverte til $p(x) = -3 + 2x - x^2 + 5x^3$.

$$\begin{aligned}p'(x) &= (-3)' + 2(x)' + (-1)(x^2)' + 5(x^3)' \\ &= 0 + 2 \cdot 1 - 1(2x) + 5(3x^2)\end{aligned}$$

$$p'(x) = \underline{2 - 2x + 15x^2}$$



Finn likningen $y = ax + b$ som beskriver tangentlinjen og normallinjen gjennom $(2, \frac{1}{2})$ til $y = \frac{1}{x}$

$$\frac{-1}{x^2} = \left(\frac{1}{x}\right)'$$

Stigningstallet til tangentlinjen er

$$\left(\frac{1}{x}\right)'\Big|_{x=2} = \frac{-1}{x^2}\Big|_{x=2} = \underline{\frac{-1}{4}}$$

$$y = \frac{-1}{4}(x-2) + \frac{1}{2}$$

$$= -\frac{x}{4} - \frac{-2}{4} + \frac{1}{2} = \underline{\frac{-x}{4} + 1}$$

Normallinjen:

Stigningstallet er $\frac{-1}{-1/4} = 4$

$$y = 4(x-2) + \frac{1}{2} = \underline{4x - 7.5}$$

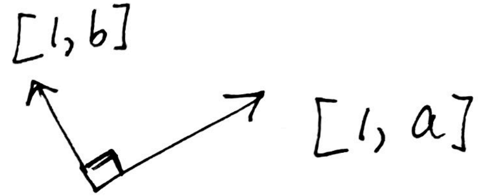
$$\left(3x^2 - 5x^6 + x^9/3\right)'$$

$$3\cdot 2x - 5\cdot 6x^5 + \tfrac{1}{3}\, 9x^8 = 6x - 30x^5 + 3x^8.$$

Normallinjer

Stigningstallet $\downarrow$ til normallinjen

$\dfrac{-1}{f'(x)}$ (vinkelrett på tangentlinjen)

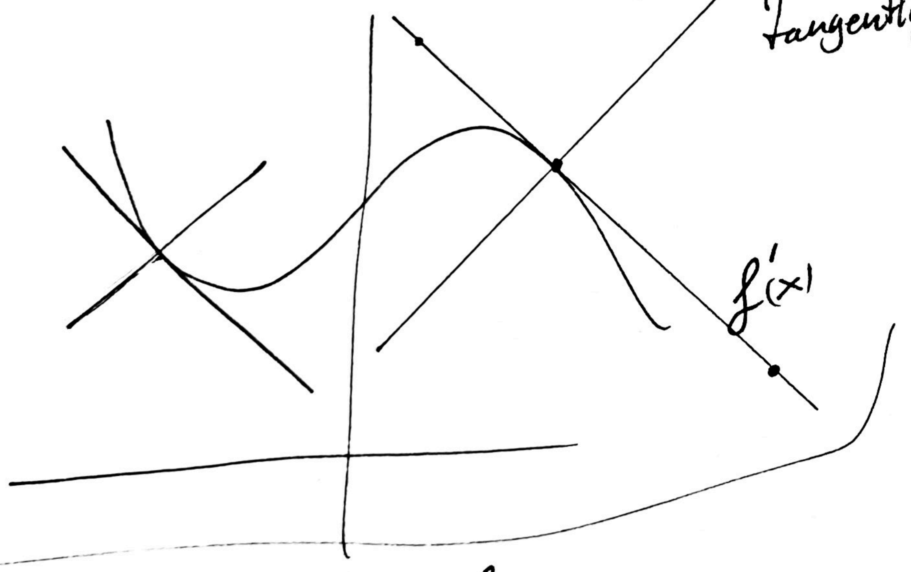


Vinkelrette:

$$[1, b]\cdot[1, a] = 0$$

$$1 + b\cdot a = 0$$

$$\underline{b = \frac{-1}{a}}$$



Eks. Finn normallinjen til

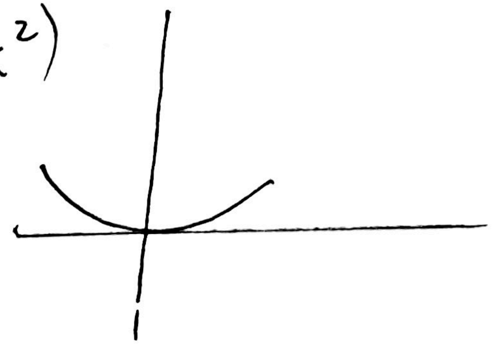
$$p(x) = 3x - x^3 - 1$$

i $x = 1$.

$$p'(x) = 3 - 3x^2 = 3(1 - x^2)$$

$$p(1) = 3 - 1 - 1 = 1.$$

$$p'(1) = 0$$



normallinjen er $\underline{x = 1}$

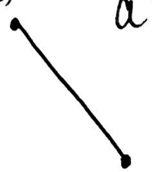
6.61 $f(x) = -x^2 + 2x + 1$

$f(1) = -1 + 2 + 1 = 2$

$f(3) = -9 + 6 + 1 = -2.$

a) $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{-2 - (2)}{2} = \underline{-2}$

b) $(1,2)$ $a = -2$



$y = a(x - x_0) + y_0$

$y = -2(x - 1) + 2$

$y = \underline{-2x + 4}$

6.73 $f(x) = x^3 - 4x$

a) $f'(x) = 3x^2 - 4$ $f'(1) = 3 - 4 = \underline{-1}$

$f(1) = 1 - 4 = -3.$

b) Tangenta i $x = 1$ går gjennom $(1, f(1)) = (1, -3)$ og har stigningstall $f'(1) = -1.$

$y = (-1)(x - 1) + -3$

$\underline{y = -x - 2}$

c) Stigningstall til normalen $\dfrac{-1}{f'(1)} = 1.$

$y = 1(x - 1) + (-3)$

$y = \underline{x - 4}$

Alternativ notasjon for den deriverte

$f'(x) = \frac{df}{dx}$ Leibnitz notasjonen.

$g(x) = (2+x)^3$

$= f(u(x))$.

$2+x = u(x)$

$f(u) = u^3$

$g'(x) = \frac{df}{du} \cdot \frac{du}{dx}$

$= 3u^2 \cdot 1$

$((2+x)^3)' = \underline{3(2+x)^2}$

Ganger ut:

$g(x) = 2^3 + 3\cdot 2^2 x + 3\cdot 2x^2 + x^3$

$g'(x) = 3\cdot 4 \cancel{x} + 3\cdot 2\cdot 2x + 3x^2$

$= 3(4 + 4x + x^2)$

$= 3(2+x)^2$ ✓

Opp. Deriver $(3-4x)^5$.

$((3-4x)^5)' = 5(3-4x)^4 \cdot (-4)$

$= \underline{-20(3-4x)^4}$

$f(u) = u^5$, $f'(u) = 5u^4$

$u(x) = 3-4x$, $u'(x) = -4$

$$(6x^5)' = 6\underbrace{(x^5)'}_{x^n} = 6(\underbrace{5x^{5-1}}_{n x^{n-1}})$$

$$= 6 \cdot 5 \cdot x^4$$

$$= \underline{30x^4}.$$

$$(x^n)' = n x^{n-1}$$

$$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$$

$$(x^{-1})' = (-1)\,x^{-1-1} = (-1) \cdot x^{-2} = -1 \cdot \frac{1}{x^2} = \frac{-1}{x^2}.$$

$$= -x^{-2}$$

Følger regneregelen...

Fra definisjonen

$$\left(\frac{1}{x}\right)' = \lim_{h\to 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} = \lim_{h\to 0} \frac{\left(\frac{x-(x+h)}{x(x+h)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{-h}{x(x+h)}\right)}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \underline{\frac{-1}{x^2}}$$

$$\left(\frac{1}{3x+1}\right)' = \left(f(u(x))\right)' = f'(u(x)) \cdot u'(x)$$

$$f(u) = \frac{1}{u} \qquad f'(u) = (-1)u^{-1-1} = \frac{-1}{u^2}$$

$$u(x) = 3x+1 \qquad u'(x) = 3$$

$$\left(\frac{1}{3x+1}\right)' = \frac{-1}{(3x+1)^2} \cdot (3) = \frac{-3}{(3x+1)^2}$$

ytrefunksjon $\quad f(u) = \frac{1}{u}, \quad f'(u) = \frac{-1}{u^2}$

kjerne $\quad u(x) = 4-x, \quad u'(x) = -1$

$$= 4+(-1)x$$

oppg. $\left(\frac{1}{4-x}\right)'$

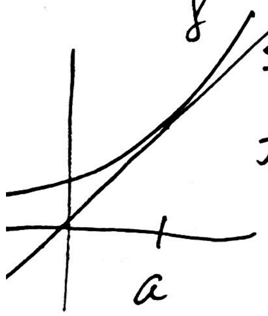
$$= \frac{-1}{(4-x)^2} \cdot (-1) = \frac{1}{(4-x)^2}$$

Derivasjon

Lineær:

$\left(f(x)+g(x)\right)' = f'(x) + g'(x)$

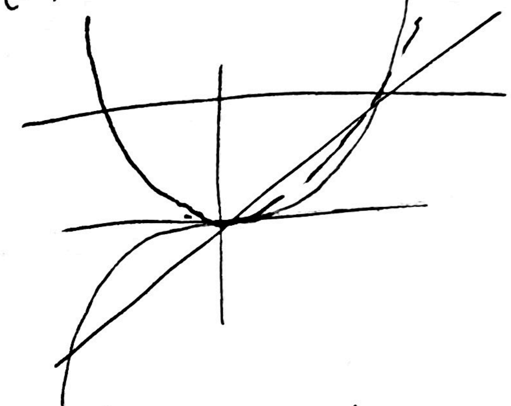
$\left(k\cdot f(x)\right)' = k\cdot f'(x)$ k konstant



stigningstall $f'(a)$

$(k)' = 0$

$(x)' = 1$



$(x^2)'$ $\cancel{(x^2)}$ $= 2x$

$(x^3)' = 3x^2$

etc. generelt $\underline{(x^n)' = nx^{n-1}}$

$$(x^{-1})' = \left(\frac{1}{x}\right)' = \frac{-1}{x^2} = (-1)\,x^{-2}.$$

Lineær kjerneregel

$$\left(f(ax+b)\right)' = f'(ax+b)\cdot\overbrace{(ax+b)'}^{a}$$

$u = ax+b$ kjerne

f ytre funksjon

$$\frac{d f(u(x))}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$$

$$\left(f(u(x))\right)' = \lim_{h\to 0} \left(\frac{f(a(x+h)+b) - f(ax+b)}{h} \right) \cdot \frac{a}{a}$$

$u(x) = ax+b$

$a \neq 0$

$(a(x+h)+b) - (ax+b) = a \cdot h$

$$\left(f(u(x))\right)' = \underbrace{\lim_{h\to 0} \frac{f(u(x)+ah) - f(u(x))}{ah}}_{f'(u(x))} \cdot a$$

$$\underline{\left(f\,u(x)\right)' = f'(u(x)) \cdot u'(x)}$$ for $u(x) = ax+b$.

også gyldig for $a=0$.

$$\left(\sqrt{x}\right)' = \left(x^{1/2}\right)' = \frac{1}{2}x^{\frac{1}{2}-1} = \underline{\frac{1}{2\sqrt{x}}}$$

Fra definisjonen av den deriverte

$$\left(\sqrt{x}\right)' = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$(b-a)(b+a) = b^2 - a^2$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{\overbrace{(x+h)-x}^{h}}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \underline{\underline{\frac{1}{2\sqrt{x}}}}$$

Deriver $p(x) = 2x + \frac{1}{4}x^4 - 3x^2 + \pi$

($\pi \sim 3.14\ldots$ tall)

$$p'(x) = (2x)' + \frac{1}{4}(x^4)' - 3(x^2)' + (\pi)'$$

$$= 2 + \frac{1}{4}(4x^3) - 3(2x) + 0$$

$$p'(x) = 2 + x^3 - 6x = \underline{x^3 - 6x + 2}$$

f ytre funksjon

u kjernen

$$\big(f(u(x))\big)' = f'(u(x)) \cdot u'(x)$$

$$\big(f(ax+b)\big)' = \underline{f'(ax+b) \cdot a}$$

$$\left((1-2x)^5\right)' = 5(1-2x)^4 \cdot (-2)$$

$$= \underline{-10(1-2x)^4}$$

$$(y^5)' = 5y^4$$

$$(1-2x)' = (1)' + (-2)(x)'$$

$$= \underline{-2}$$

Hva er $\left((2+3x)^7\right)'$

Vi kan gange ut og så derivere leddvis.

$f(ax+b)$

sammensatt funksjon

$f(y) = y^7$

$f(2+3x) = (2+3x)^7$

Lineær substitusjon

Kjerneregel for lineær substitusjon

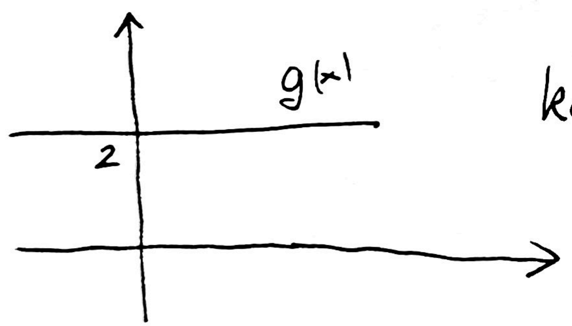
$$\left(f(ax+b)\right)' = f'(ax+b)\cdot(ax+b)'$$
$$= a\cdot f'(ax+b)$$

<u>Eks</u>

$$\left((2+3x)^7\right)' = 7(2+3x)^6 \cdot \underbrace{(2+3x)'}_{3}$$
$$= 7\cdot 3(2+3x)^6$$
$$= \underline{\underline{21(2+3x)^6}}$$

Vi kan derivere alle polynomer

$$\left(-3 + 5x^4 - 7x^7\right)' = (-3)' + 5(x^4)' + (-7)(x^7)'$$
$$= 0 + 5(4x^3) + (-7)(7x^6)$$
$$= \underline{20x^3 - 49x^6}$$



konstant funksjon

$g(x) = 2$ for alle $x \in D_g$

$$\frac{g(x+h) - g(x)}{h} = \frac{2-2}{h} = 0$$

så $g'(x) = 0$ for alle $x \in D_g$

Hva må $g(x)$ være hvis $g'(x) = 0$?

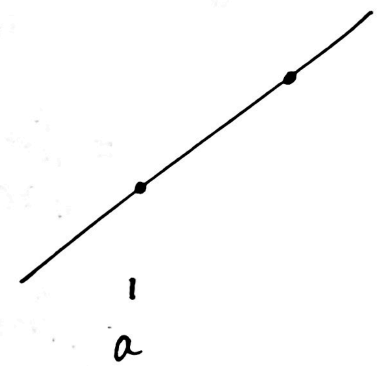
Hvis definisjonsmengden er ett intervall, då må $g(x)$ være en konstant funksjon.

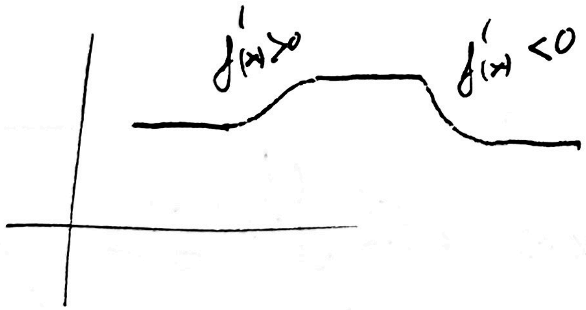
$f(x) = 2x - 1$

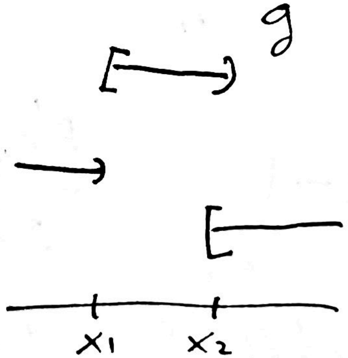
Fra definisjonen er $f'(a)$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{2(a+h) - 1 - (2a-1)}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = \underline{2}$$



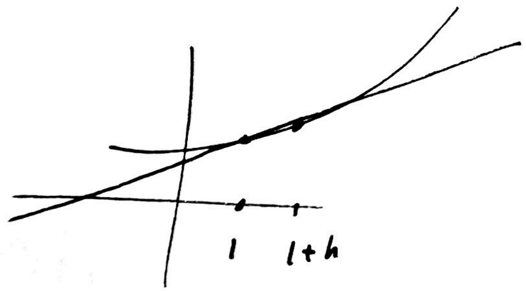




$g(x)$ er ikkje kont. i x_1 og x_2.
($\Rightarrow$ ikkje deriverbar)

$f'(x) = 0$ i ett intervall
$\Rightarrow f(x) = k$ konstant

6.63 $f(x) = x^2 + 2x - 1$ a) $f'(1)$ b) $f'(0)$ ved regning.

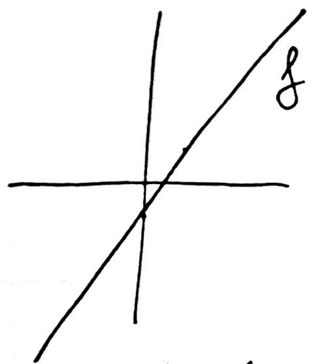


$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{((1+h)^2 + 2(1+h) - 1) - (1 + 2 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{((1+h)^2 - 1) + 2(1+h-1) - 1 - (-1)}{h}$$

$$= \lim_{h \to 0} \frac{2h + h^2 + 2h + 0}{h} = \lim_{h \to 0} (2+h) + 2 = \underline{\underline{4}}$$



6.62 a) $f(x) = 2x - 1$

a) $f'(x) = 2$
så $f'(3) = 2$

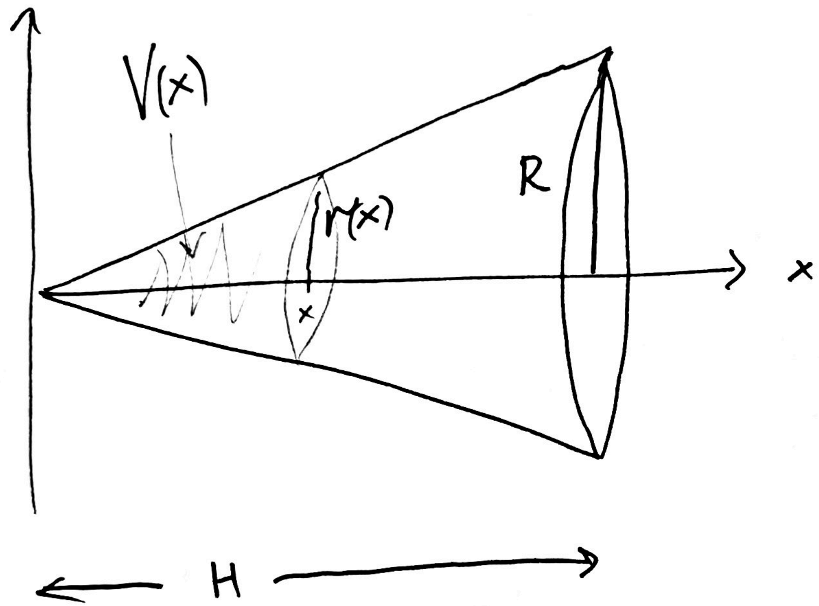
b) Momentan vekstfart ~~til~~ $f(x)$ når $x = 3$ er lik $f'(3) = \underline{\underline{2}}$

c) Grafen til $f(x)$ er en linje, som også er lik tangentlinjen i hvert punkt på grafen.

En funksjon $F(x)$ slik at $F'(x) = f(x)$ kalles en <u>antiderivert</u> til $f(x)$

Det <u>ubestemte integralet</u> til $f(x)$ er samlingen av alle antideriverte til $f(x)$

Det skrives som $\int f(x)\,dx = F(x) + k$



$V(H)$ = volum til kjegle med radius R og høyde H

$$V(H) = \frac{\pi R^2}{H^2} \cdot \frac{H^3}{3} = \underline{\underline{\frac{\pi}{3} R^2 H}}$$

$$\frac{r(x)}{x} = \frac{R}{H}$$

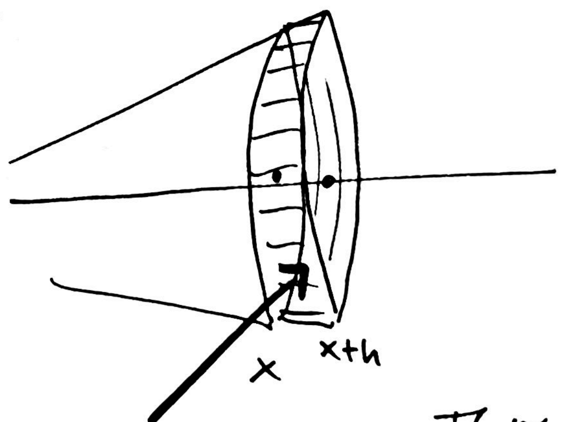
$$r(x) = \frac{R}{H} \cdot x$$

$$V'(x) = \pi r(x)^2$$
$$= \pi\left(\frac{R}{H} x\right)^2 = \frac{\pi R^2}{H^2} \cdot x^2$$

Så $V(x) = \frac{\pi R^2}{H^2} \cdot \frac{x^3}{3} + k$

$V(0) = 0 \Rightarrow k = 0$ så

$$\underline{V(x) = \frac{\pi R^2}{H^2} \cdot \frac{x^3}{3}}$$



$$\frac{V(x+h) - V(x)}{h} \sim \frac{\pi r(x)^2 \cdot h}{h} = \pi r(x)^2$$

Bestem $f(x)$ når

a) $f'(x) = 3x^2$: $f(x) = x^3 + k$

b) $f'(x) = x^2$. : $f(x) = \frac{1}{3}x^3 + k$.

c) $f'(x) = x^{-2}$: $f(x) = \frac{-1}{x} + k$

a konst

$(ax^3)' = a(x^3)' = a \cdot 3x^2$ Dette er lik x^2 når $a \cdot 3 = 1$: $a = \frac{1}{3}$

d) $f'(x) = 3x^4$: $f(x) = \frac{3}{5}x^5 + k$

$(x^5)' = 5x^4$

$(kx^5)' = \underbrace{k \cdot 5}_{3} \cdot x^4$ $k \cdot 5 = 3$

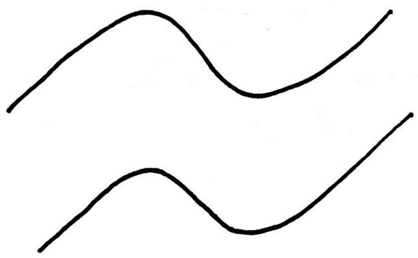
$k = 3/5$

$k = 3/5$: $\left(\frac{3}{5}x^5\right)' = \frac{3}{5} \cdot 5x^4 = 3x^4$.

$f'(x) = 0$ på et intervall $\Rightarrow$ $f(x) = k$ konstant funksjon

$f'(x) = g'(x)$ ———— $\Rightarrow$ $f(x) = g(x) + k$.

$\Leftrightarrow (f(x) - g(x))' = 0$



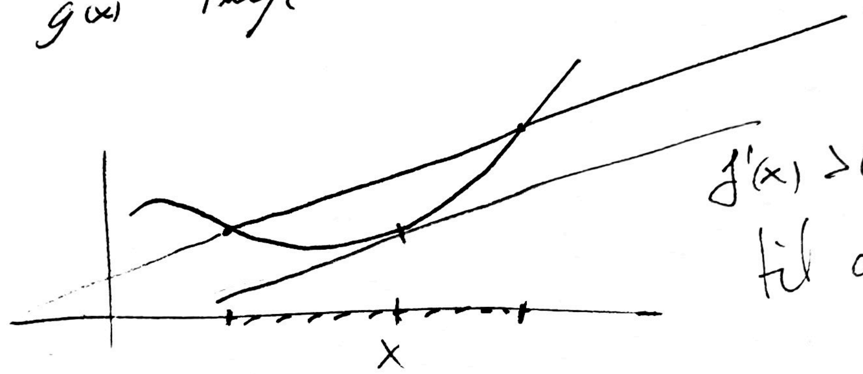
$f'(x) = 2x$ siden $(x^2)' = 2x$

$\Rightarrow f(x) = x^2 + k$ k konstant

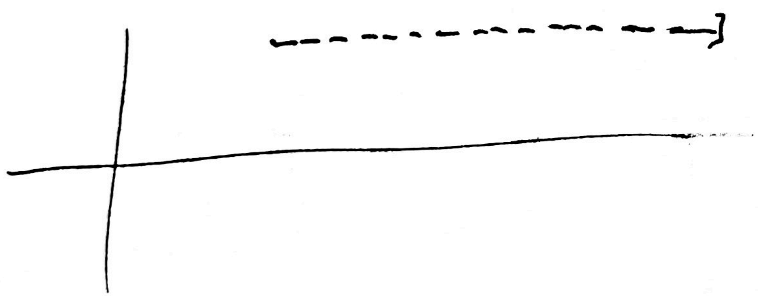
$s'(t) =$

start $V(t)$ kjent slutt

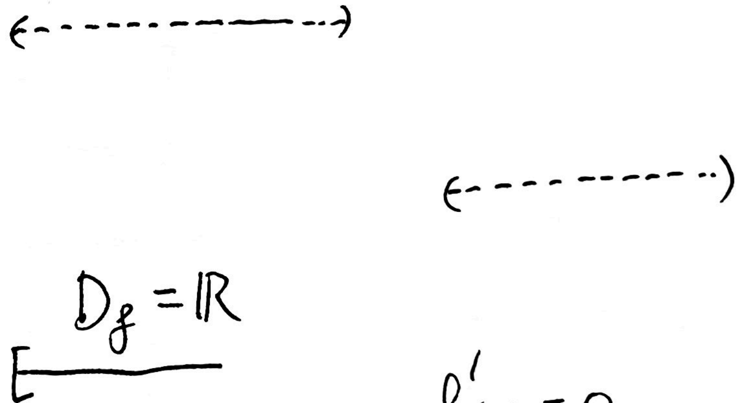
Anta $g(x)$ ikkje er konst



$f'(x) > 0$ motsigelse til at $f'(x) = 0$ for alle x.



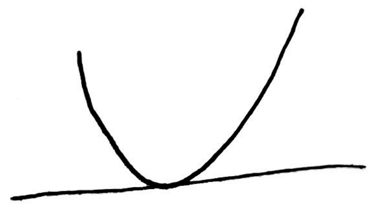
$D_f = \mathbb{R}$

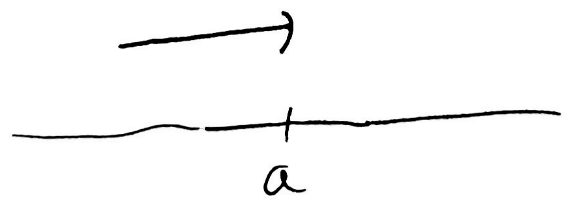


$f'(x) = 0$

ikkje deriverbar i $x = a$

$(x^2)' = 2x$ alle x



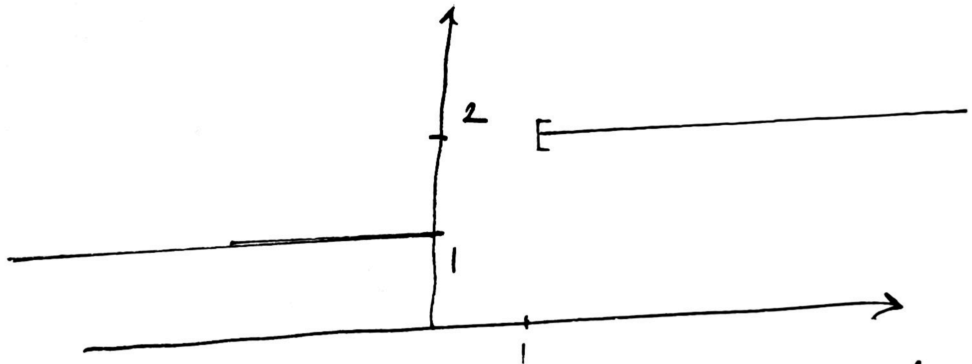


$D_{f'} = \underline{\mathbb{R} \setminus \{a\}}$

Eks

$$\left(3 - 5x + 3x^2 - 8x^4\right)' = (3)' + (-5x)' + (3x^2)' + (-8x^4)'$$
$$= (3)' + (-5)(x)' + 3(x^2)' + (-8)(x^4)'$$
$$= 0 + (-5)\cdot 1 + 3(2x) + (-8)(4x^3)$$
$$= \underline{-5 + 6x - 32x^3}$$

Eks $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 1 \end{cases}$



$f'(x) = 0$ i D_f

$f(x)$ er konstant på hver komponent av definisjonsmengden (adskilte intervaller)