Mandag
25.01.2021

7.6 Kjerneregelen.

$f(u(x))$ "først u så f"

$= f \circ u(x)$ alternativ skrivemåte.

Ytre funksjon $f(u(x))$ kjerne

$f(x) = x^2$ $\quad u(x) = 2x - 5$

$f \circ u(x) = f(u(x)) = (2x-5)^2$

$u(f(x)) = 2x^2 - 5$

typisk $f \circ u \neq u \circ f$

$g(x) = (x^3+1)^5$

$= f \circ u(x)$ hvor $u(x) = x^3 + 1$

$f(x) = x^5$

$g(x)$ er en sammensetning av to enkle funksjoner

$g(x) = \sqrt{1 + x^3}$ $\quad u(x) = 1 + x^3$, $f(x) = \sqrt{x}$ $\quad g(x) = f \circ u(x)$

alternativt: $\quad v(x) = x^3$, $h(x) = \sqrt{1 + x}$ $\quad g(x) = h \circ v(x)$

$g(x) = \sqrt{2 + (1-x)^5}$ $\quad f(x) = \sqrt{x}$

$2 + (1-x)^5$

$= f \circ h \circ u(x)$ $\quad h(x) = 2 + x^5$

$u(x) = (1 - x)$

$g(x) = \cos^3(x) = (\cos(x))^3$ $\quad u(x) = \cos x$

$= f \circ u(x)$ $\quad f(x) = x^3$.

$f(x) = \sqrt{x}$ $\quad u(x) = x - 3$

$D_f = [0, \infty\rangle$ $\quad D_u = \mathbb{R}$

$f \circ u(x) = \sqrt{x-3}$ $\quad D_{f \circ u} = [3, \infty\rangle$

$u \circ f(x) = \sqrt{x} - 3$ $\quad D_{u \circ f} = [0, \infty\rangle$

<u>oppg</u>

$f(x) = \frac{1}{x-2}$ $\quad D_f = \mathbb{R} \setminus \{2\}$ $\quad u(x) = x^2$ $\quad D_u = \mathbb{R}$

$f \circ u(x) = f(x^2) = \frac{1}{x^2 - 2}$ $\quad D_{f \circ u} = \underline{\mathbb{R} \setminus \{\pm\sqrt{2}\}}$

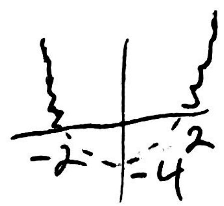
alle $x \in D_u = \mathbb{R}$ slik at $u(x) = x^2 \in D_f = \mathbb{R} \setminus \{2\}$

oppg.

$f(x) = \sqrt{x}$ $\quad u(x) = x^2 - 4$

$f(u(x)) = f \circ u(x) = \sqrt{x^2 - 4}$ $\quad D_{f \circ u} = \underline{\langle -\infty, -2] \cup [2, \infty\rangle}$

alle x s.a $u(x) = x^2 - 4$ er i $D_f = [0, \infty\rangle$.



Kjerneregelen $\quad (f \circ u(x))' = f'(u(x)) \cdot u'(x)$

eksempler: $\quad (x^3+1)^5 = f \circ u(x)$ hvor $f(x) = x^5$, $u(x) = x^3+1$

$$((x^3+1)^5)' = (f \circ u(x))'$$
$$= f'(u(x)) \cdot u'(x)$$
$$= 5(u(x))^4 \cdot 3x^2$$
$$= 5(x^3+1)^4 \cdot 3x^2$$
$$= \underline{15 \cdot x^2 (x^3+1)^4}$$

$$f'(x) = 5x^{5-1} = 5x^4$$
$$u'(x) = (x^3)' + (1)'$$
$$= 3x^{3-1} + 0$$
$$= 3x^2$$

Mindre detaljer: $\quad ((x^3+1)^5)' = 5(x^3+1)^4 \cdot (x^3+1)'$
$$= 5(x^3+1)^4 \cdot 3x^2$$
$$= \underline{15(x^3+1)^4 \cdot x^2}$$

Oppg. Deriver $\sqrt{1+x^3} = (1+x^3)^{1/2}$

$\left((\sqrt{x})' = \frac{1}{2\sqrt{x}} \qquad (1+x^3)' = 3x^2 \right)$

$$\left(\sqrt{1+x^3}\right)' = \frac{1}{2\sqrt{1+x^3}} \cdot (1+x^3)'$$

$$= \underline{\frac{3x^2}{2\sqrt{1+x^3}}}$$

$$\left(\sqrt[3]{x}\right)' = \left(x^{1/3}\right)' = \frac{1}{3} x^{1/3 - 1} = \frac{1}{3} x^{-2/3}$$

$$= \frac{1}{3 \cdot x^{2/3}} = \underline{\frac{1}{3\sqrt[3]{x^2}}}$$

bruker $(x^r)' = r x^{r-1}$ $\quad$ $\underline{r \text{ reelt tall}}$

Deriver $\sqrt{2+(1-x)^5}$

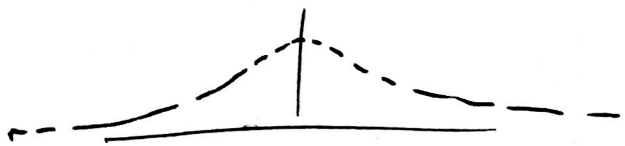
$$\left(\sqrt{2+(1-x)^5}\right)' = \frac{1}{2\sqrt{2+(1-x)^5}} \cdot \underbrace{\left(2+(1-x)^5\right)'}_{\left((1-x)^5\right)'}$$

bruker kjerneregelen en gang til

$$= \frac{1}{2\sqrt{2+(1-x)^5}} \cdot 5(1-x)^4 \cdot \underbrace{(1-x)'}_{-1}$$

$$\left(\begin{aligned} &1-x \\ &= 1+(-1)\cdot x \ldots \end{aligned} \right)$$

$$= \frac{-5\cdot(1-x)^4}{2\sqrt{2+(1-x)^5}}$$



oppg

Deriver: $\frac{1}{1+x^2} = (1+x^2)^{-1}$

$= \frac{1}{u(x)} = u(x)^{-1}$ når $u(x) = 1+x^2$.

$$\left((1+x^2)^{-1}\right)' = \frac{-1}{(1+x^2)^2} \cdot (1+x^2)'$$

$$= \frac{-2x}{(1+x^2)^2}$$

$$\left(\begin{aligned} \left(\frac{1}{x}\right)' = \left(x^{-1}\right)' &= -1 \cdot x^{-1-1} \\ &= -1 \cdot x^{-2} \\ &= \frac{-1}{x^2} \end{aligned} \right)$$

$$\left(\begin{aligned} &(1+x^2)^2 \\ &= (1+x^2)(1+x^2) \\ &= 1+2x^2+x^4 \end{aligned} \right)$$

$$f(x) = \left(2 + \sqrt{3 - x^5}\right)^7$$

$$f'(x) = 7\left(2 + \sqrt{3 - x^5}\right)^6 \cdot \underbrace{\left(2 + \sqrt{3 - x^5}\right)'}_{\left(\sqrt{3 - x^5}\right)'}$$

$$= 7\left(2 + \sqrt{3 - x^5}\right)^6 \cdot \frac{1}{2\sqrt{3 - x^5}} \cdot \underbrace{\left(3 - x^5\right)'}_{(-5x^4)}$$

$$f'(x) = 7\left(2 + \sqrt{3 - x^5}\right)^6 \frac{-5x^4}{2\sqrt{3 - x^5}}$$

$$= \frac{-35x^4\left(2 + \sqrt{3 - x^5}\right)^6}{2\sqrt{3 - x^5}}$$

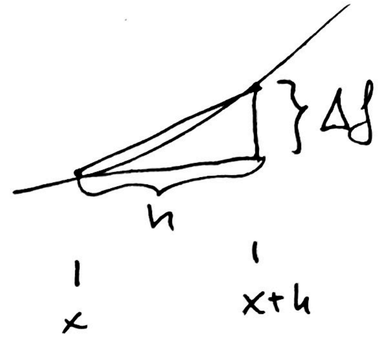
$$\left(\sqrt{x}\right)' = \left(x^{1/2}\right)'$$

$$= \left(\frac{1}{2}\right) \cdot x^{\frac{1}{2} - 1}$$

$$= \frac{1}{2} x^{-1/2} = \frac{1}{2}\left(x^{1/2}\right)^{-1}$$

$$= \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$$\Delta f = f(x+h) - f(x)$$

$$(f \circ u(x))' = \lim_{h \to 0} \frac{f(u(x+h)) - f(u(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(u(x+h)) - f(u(x))}{u(x+h) - u(x)} \cdot \frac{u(x+h) - u(x)}{h}$$

$u(x+h) - u(x)$
$\neq 0$
$h \neq 0$
liten

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$h = \Delta x$

Leibnitz notasjon for den derivert $f'(x) = \frac{df}{dx}$

$$(f\,u(x))' = \frac{d\,f \circ u}{dx} = \frac{df(u)}{du} \circ \frac{du}{dx}$$

kjerneregelen

Deriver: $f(x) = \sqrt{1+\sqrt{2+\sqrt{x}}}$ $\qquad \left((\sqrt{x})' = \frac{1}{2\sqrt{x}} \right)$

$$f'(x) = \frac{1}{2\sqrt{1+\sqrt{2+\sqrt{x}}}} \left(1+\sqrt{2+\sqrt{x}}\right)'$$

$$\left(0 + \frac{1}{2\sqrt{2+\sqrt{x}}}\right) \cdot \underbrace{\left(2+\sqrt{x}\right)'}_{0+\frac{1}{2\sqrt{x}}}$$

$$= \frac{1}{2\sqrt{1+\sqrt{2+\sqrt{x}}}} \cdot \frac{1}{2\sqrt{2+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{8\sqrt{1+\sqrt{2+\sqrt{x}}} \cdot \sqrt{2+\sqrt{x}} \cdot \sqrt{x}}$$

$$(x^r)^s = x^{r \cdot s}$$ sammensatt funksjon.

Direkte: $(x^{r \cdot s}) = \underline{r \cdot s\, x^{r \cdot s - 1}}$

Kjerneregel: $((x^r)^s)' = s(x^r)^{s-1} \cdot (x^r)' = s(x^r)^{s-1} \cdot r x^{r-1}$

$$= s \cdot r\, x^{r(s-1)+r-1} = s \cdot r\, x^{r \cdot s - 1}$$ ✓

Bevis for kjerneregelen.

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

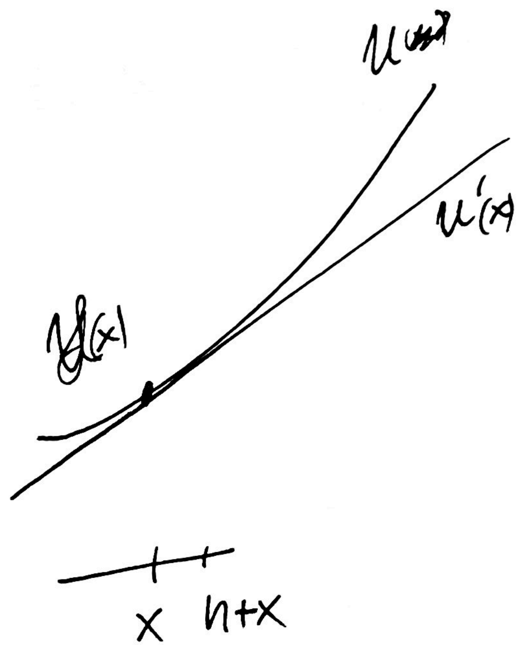
$$\Leftrightarrow \lim_{h \to 0} \left(\frac{u(x+h) - u(x)}{h} - u'(x) \cdot \frac{h}{h} \right) = 0$$

$$\Leftrightarrow \lim_{h \to 0} \frac{u(x+h) - u(x) - u'(x) \cdot h}{h} = 0$$

La $u(x+h) - u(x) - u'(x) \cdot h = r_x(h)$

$$u(x+h) = u(x) + u'(x) \cdot h + r_x(h)$$

og $\lim_{h \to 0} \frac{r_x(h)}{h} = 0$



$$(f \circ u(x))' = \lim_{h \to 0} \frac{f \cdot (u(x+h)) - f(u(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(u(x) + u'(x) \cdot h + r(h)) - f(u(x))}{h}$$

$$f(u(x) + k) = f(u(x)) + f'(u(x)) \cdot k + S(k) \quad \Big| \; \frac{S(k)}{k} \to 0 \text{ nếu } k \to 0$$

Là $k = u'(x) h + r(h)$

$$(f u(x))' = \lim_{h \to 0} \frac{f(u(x)) + f'(u(x)) \cdot (u'(x) \cdot h + r(h)) + S(k) - f(u(x))}{h}$$

$$= \lim_{h \to 0} \frac{f'(u(x)) \cdot u'(x) \cdot h + f'(u(x)) \cdot r(h) + S(u'(x) \cdot h + r(h))}{h}$$

$$= \lim_{h \to 0} f'(u(x)) \cdot u'(x) + \underbrace{\frac{f'(u(x)) \cdot r(h)}{h}}_{\downarrow \; 0} + \underbrace{\frac{S(u'(x) \cdot h + r(h))}{h}}_{\downarrow \; 0}$$

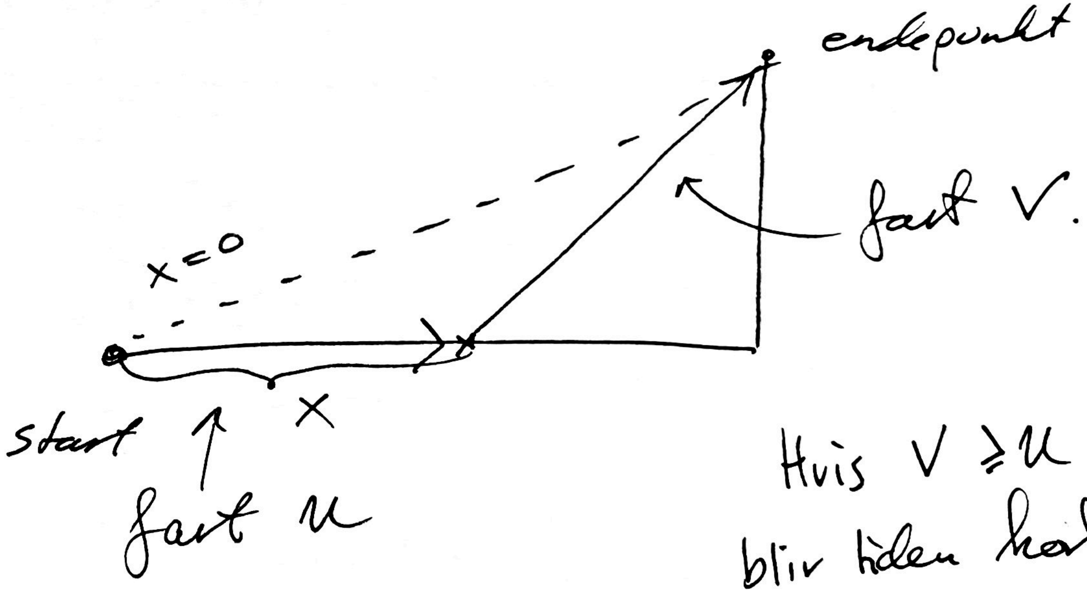
$$= f'(u(x)) \cdot u'(x)$$

Linear kjerneregel

$$\left(f(ax+b)\right)' = f'(ax+b) \cdot (ax+b)'$$
$$= a\, f'(ax+b)$$ ✓

oppgave.
Gjennomgang 17:30.

Vi ønsker å minimere tiden det tar å komme seg fra start til endepunkt.



Hvis $v \geq u$ blir tiden kortest ved $x = 0$.

Skissere grafen til $T(x)$

$V = 1$ parametre u, H

$L = 1$

tegner opp $T(x) = \dfrac{x}{u} + \dfrac{\sqrt{H^2 + (L-x)^2}}{V}$ i geogebra.

$$0 \leq x \leq 1 = L$$

$$T'(x) = \frac{1}{u} + \frac{1}{V} \frac{1}{2\sqrt{H^2 + (1-x)^2}} \cdot \left(H^2 + (L-x)^2 \right)'$$

$$= \frac{1}{u} + \frac{1}{V} \frac{1}{2\sqrt{H^2 + (L-x)^2}} \cdot \left(0 + 2(L-x) \cdot \underbrace{(L-x)'}_{-1} \right)$$

$$T'(x) = \frac{1}{u} + \left(\frac{1}{V}\right) \frac{-(L-x)}{\sqrt{H^2 + (L-x)^2}} = 0$$

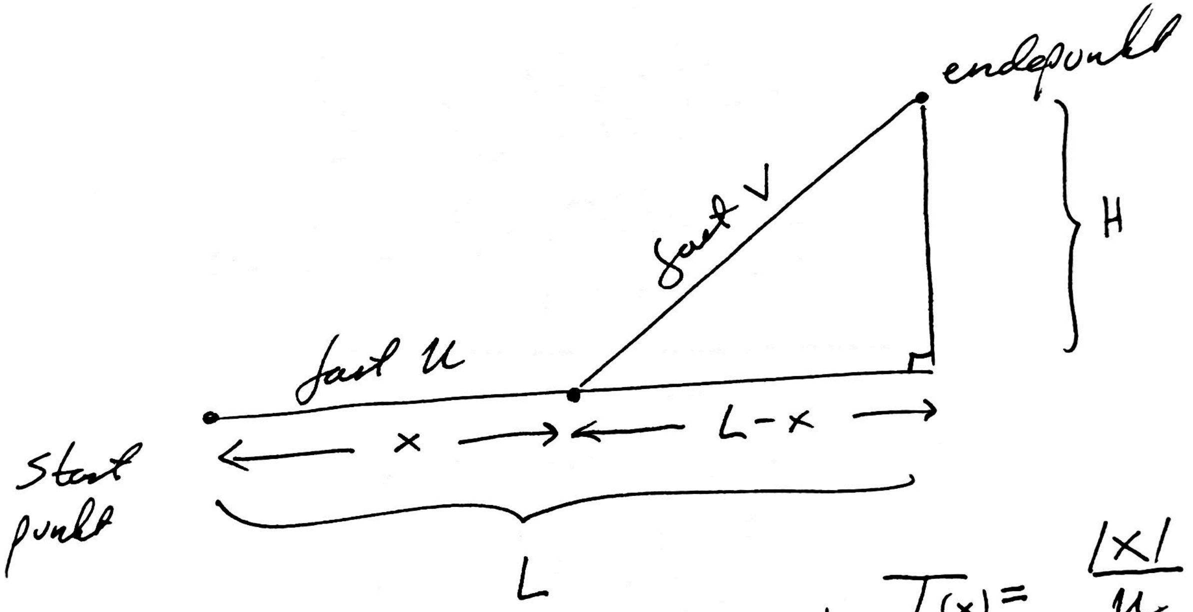
$$\Leftrightarrow \quad \frac{1}{u} = -\frac{-(L-x)}{V\sqrt{H^2 + (1-x)^2}} = \frac{(L-x)}{V\sqrt{H^2 + (L-x)^2}}$$

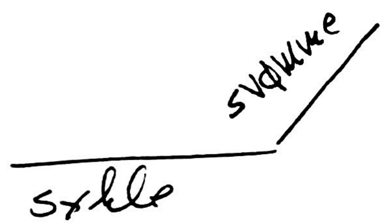
$$\underbrace{\sqrt{H^2 + (L-x)^2}}_{\geq 0} = \underbrace{\frac{u}{V}(L-x)}_{\geq 0} \qquad 0 \leq x \leq L$$

kvadrerer begge sider.

$$\Leftrightarrow \quad H^2 + (L-x)^2 = \left(\frac{u}{V}\right)^2 (L-x)^2.$$

Minimer tiden det tar å bevege seg fra start til endepunkt





$V \geq u$: $x = 0$

$u >> v$ (u mye større enn v) $x \sim L$

Hvor stor må $\frac{u}{v}$ være for at $x > 0$?

$$T(x) = \frac{|x|}{u} + \frac{\sqrt{H^2 + (L-x)^2}}{v}$$

Kortest tid for en $x \in [0, L]$.

$$T(x) = \frac{x}{u} + \frac{\sqrt{H^2 + (L-x)^2}}{v}$$

$x \in [0, L]$.

$$H^2 = \left(\frac{u}{v}\right)^2 (L-x)^2 - (L-x)^2$$

$$= \left(\left(\frac{u}{v}\right)^2 - 1\right)(L-x)^2 \qquad \frac{u}{v} > 1$$

$$H = \sqrt{\left(\frac{u}{v}\right)^2 - 1}\,(L-x) \qquad \text{deler med } L$$

$$\frac{H}{L} = \sqrt{\left(\frac{u}{v}\right)^2 - 1}\,(\underline{1-x}) \qquad 0 \leq x \leq 1$$

$$1 - x = \frac{H/L}{\sqrt{(u/v)^2 - 1}}$$

$x = 0$

$$\frac{H}{L} = \sqrt{\left(\frac{u}{v}\right)^2 - 1} \iff \left(\frac{H}{L}\right)^2 = \left(\frac{u}{v}\right)^2 - 1$$

$$\underline{\left(\frac{u}{v}\right)^2 = \left(\frac{H}{L}\right)^2 + 1}$$

$\frac{u}{v} \leq \sqrt{\left(\frac{H}{L}\right)^2 + 1}$: $x = 0$

$\frac{u}{v} > \sqrt{\left(\frac{H}{L}\right)^2 + 1}$ korlest tid når $1 - x = \frac{H/L}{\sqrt{(u/v)^2 - 1}}$

$$\underline{x = 1 - \frac{H/L}{\sqrt{(u/v)^2 - 1}}}$$

7.60 d) 3^{3x+1} $\qquad f(u) = 3^u$

* $= f(u(x))$ $\qquad u(x) = 3x+1$

* $3^{3(x+1/3)} = (3^3)^{(x+1/3)} = 27^{(x+1/3)}$

$f(u) = 27^u$ $\qquad u(x) = x + 1/3$

* $f(u) = 3^{u+1} = 3^u \cdot 3^1 = 3 \cdot 3^u$ $\qquad u(x) = 3x$

7.64 b) $y(x) = (x^4 + 2x^2)^3$

$= f(u(x))$

bwk Leibniz notasjonen

$f(u) = u^3$

$u(x) = x^4 + 2x^2$

kjerneregela

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d\,u^3}{du} \cdot \frac{du}{dx}$$

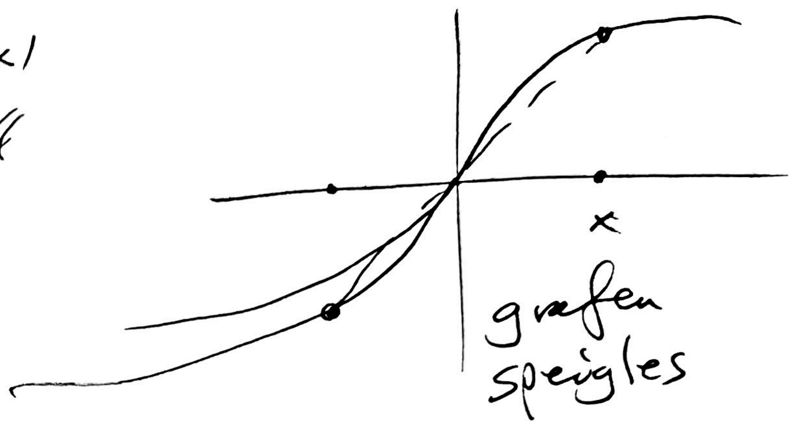
$$= \frac{du^3}{du} \cdot \frac{d(x^4 + 2x^2)}{dx} = 3u^2 \cdot (4x^3 + 2 \cdot 2x)$$

$$= 3(x^4+2x^2)^2(4x^3+4x)$$

$$= \underline{12x(x^2+1)(x^4+2x^2)^2}$$

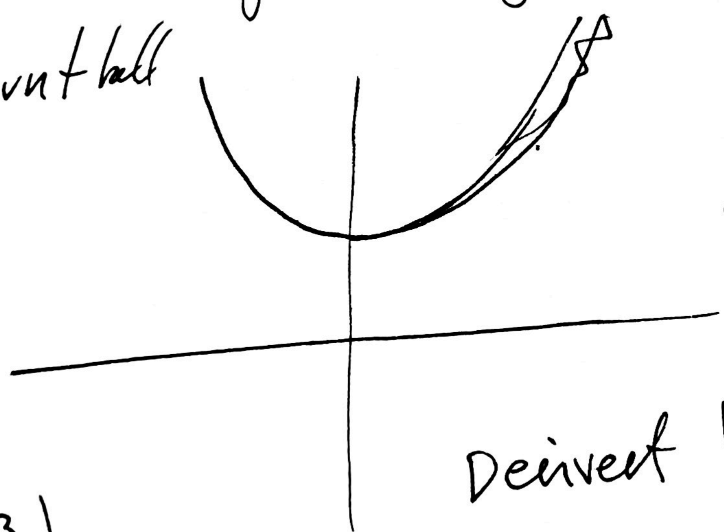
odde funksjoner $f(-x) = -f(x)$

x^n n oddetall



jevn funksjon $f(-x) = f(x)$

x^n n jevntall



grafen
spegles
om y-aksen

Derivert til odde funksjon er en jevn funksjon
— jevn — odde —

↑
bevis dette ...

$x^2 + x^3$

$= \underbrace{(x^2)}_{\text{jevn}} + \underbrace{(x^3)}_{\text{odde}}$

$\underline{f(x) = f_{odd}(x) + f_{jevn}(x)}$ ←

alle funksjoner på $[-a, a]$ er en sum av en jevn og en odde funksjon.