2 mars 2021

15.3 og 4 Antiderivasjon og ubestemte integral

$F'(x) = f(x)$ da er $F(x)$ en antiderivert til $f(x)$

$\int f(x)\,dx$ = samling av alle antideriverte til $f(x)$

$= F(x) + C$

Ubestemte integralet til $f(x)$ med hensyn til x.

$$f(x) = x^{-3} = \frac{1}{x^3}$$

$$\left(\frac{1}{x^2}\right)' = \left(x^{-2}\right)' = -2 \cdot x^{-2-1} = -2x^{-3}$$

$$-\frac{1}{2} \cdot \left(\frac{1}{x^2}\right)' = \frac{-1}{2}\left(-2x^{-3}\right) = x^{-3}$$

$$\left(-\frac{1}{2} \cdot \frac{1}{x^2}\right)' = x^{-3}$$

Så $F(x) = \frac{-1}{2} \cdot \frac{1}{x^2}$ er en antiderivert til x^{-3}

$$\int x^{-3}\,dx = \underline{\frac{-1}{2} \cdot \frac{1}{x^2} + C}$$

(en konstant på komponenten $x > 0$ og en annen hvor $x < 0$)

$$\int x^s \, dx = \begin{cases} \dfrac{x^{s+1}}{s+1} + c & s \neq -1 \\ \ln|x| + c & s = -1 \end{cases}$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

$$(e^x)' = e^x$$

$$\int e^x \, dx = e^x$$

$$\int \frac{u'(x)}{u(x)} \, dx = \ln|u(x)| + c$$

$$(e^{kx})' = e^{kx}(kx)' = k e^{kx} \qquad k \neq 0$$

$$\left(\frac{1}{k} e^{kx}\right)' = e^{kx}$$

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + c$$

$$e^{kx+b} = e^{kx} \cdot e^b$$

$$\left(\frac{1}{k} e^{kx+b}\right)' = e^{kx+b}.$$

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c$$

$$\left(\frac{3}{4}x^4\right)' = 3\frac{\left(x^4\right)'}{4} = \frac{3\cdot 4x^3}{4} = 3\cdot x^3$$

$$\int 3x^3\,dx = 3\int x^3\,dx = 3\cdot\frac{x^4}{4} + c$$

$$\int e^{-2x}\,dx = \frac{-1}{2}e^{-2x} + C$$

$$\left(\frac{-1}{2}\cdot e^{-2x}\right)' = \frac{\left(e^{-2x}\right)'}{-2} = \frac{e^{-2x}\cdot(-2x)'}{-2} = \frac{-2e^{-2x}}{-2} = e^{-2x}$$

oppg.

$$\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-1/3}\,dx = \frac{x^{2/3}}{2/3} + c = \underline{\frac{3}{2}x^{2/3} + c}$$

$$\int 3x^3 - 4x + 2\,dx = 3\cdot\frac{x^4}{4} - 4\frac{x^2}{2} + 2x + c$$

$$= \underline{\frac{3}{4}\cdot x^4 - 2x^2 + 2x + c}$$

$$\int 3\cdot e^{-2x}\,dx = 3\left(\frac{e^{-2x}}{-2}\right) + c = \underline{\frac{-3}{2}e^{-2x} + c}$$

$$\int \frac{1}{x-3}\,dx \qquad \text{La } u = x-3, \quad u' = 1$$

$$= \int \frac{u'}{u}\,dx = \ln|x-3| + c \qquad x \neq 3.$$

$$\left(e^{u(x)}\right)' = e^{u(x)}\cdot(u(x))' = u'(x)\,e^{u(x)}$$

Så $\underline{\int u'(x)\,e^{u(x)}\,dx} = \underline{e^{u(x)} + c}$

$$\int \underbrace{2x}_{u'}\cdot\underbrace{e^{x^2}}_{e^u}\,dx = e^{x^2} + c$$

$$\int 2^x \, dx \qquad \text{hint} \quad 2^x = \left(e^{\ln 2}\right)^x = e^{\ln 2 \cdot x}$$

$$\int e^{\ln 2 \cdot x} \, dx \quad \left(= \int e^{kx} \, dx \ \text{hvor } k = \ln 2 \right)$$

$$= \frac{1}{\ln 2} e^{\ln 2 \cdot x} + C$$

$$= \underline{\frac{2^x}{\ln 2} + C}$$

$$\int 10^{-x/2} \, dx = \int \left(e^{\ln 10}\right)^{-x/2} dx = \int e^{\frac{-\ln 10}{2} \cdot x} \, dx$$

$$\left(\int e^{kx} \, dx, \text{ hvor } k = \frac{-\ln(10)}{2} \right)$$

$$\int 10^{-x/2} \, dx = \frac{1}{-\ln(10)/2} 10^{-x/2} + C = \underline{\frac{-2}{\ln 10} \cdot 10^{-x/2} + C}$$

$$\int 10^{-\frac{1}{2}x}\,dx \underset{(k=-\frac{1}{2})}{=} \int 10^{kx}\,dx = \frac{1}{k}\cdot\frac{10^{kx}}{\ln 10} + c$$

$$= \frac{-2}{\ln(10)}\, 10^{-x/2} + c$$

$$10^x = \left(e^{\ln 10}\right)^x = \cancel{10}\; e^{\ln 10\cdot x}$$

så $\quad (10^x)' = e^{\ln 10\cdot x}\cdot(\ln 10\cdot x)'$

$$= \ln(10)\; 10^x$$

så $\quad \left(\dfrac{10^x}{\ln 10}\right)' = 10^x$

Linear substitusjon

$$(F(x))' = f(x)$$

$$(F(ax+b))' = \underbrace{F'(ax+b)}_{f(ax+b)} \cdot \underbrace{(ax+b)'}_{a}$$

så $$\left(\frac{1}{a} F(ax+b)\right)' = f(ax+b)$$

$$\int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C$$

$$\left(= \frac{1}{a} \int f(u)\,du \quad \text{hvor } u = ax+b \right)$$

$$\int \overbrace{(x+5)^7}^{u^7}\,dx \qquad \left(\frac{u^8}{8}\right)' = u^7$$

$$= \frac{1}{1} \cdot \frac{(x+5)^8}{8} + C = \frac{1}{8}(x+5)^8 + C$$

$$\left(F(u(x))\right)' = f(u(x)) \cdot u'(x)$$

$$\int u'(x)\, f(u(x))\, dx = F(u(x)) + C$$

$$= \int f(u)\, du \ (+ C)$$

$$\int \underbrace{2x}_{u'} \, (\underbrace{1+x^2}_{u})^7 \, dx = \int u^7 du = \frac{(1+x^2)^8}{8} + C$$

$$\int (1+x^2)^7 \, dx \quad ??? \quad \text{gange ut...}$$

oppg.

$$\int \frac{4}{(2x-3)^3}\, dx = 4\int \frac{1}{(2x-3)^3}\, dx = 4\int (2x-3)^{-3}\, dx$$

(En antiderivert til u^{-3} er $\frac{u^{-2}}{-2}$)

$$\int \frac{4}{(2x-3)^3}\, dx = 4\int (2x-3)^{-3} dx = 4\left(\frac{1}{2} \cdot \frac{(2x-3)^{-2}}{-2}\right) + C$$

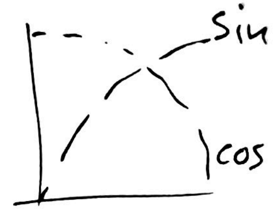
$$= -(2x-3)^{-2} + C = \underline{\frac{-1}{(2x-3)^2} + C}$$

$(\sin x)' = \cos x$

$\int \cos x \, dx = \sin x + c$

$(\cos x)' = -\sin x$

$\int \sin x \, dx = -\cos x + c$



$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

Så $\int \frac{1}{\cos^2 x} dx = \tan x + c$

$\int 1 + \tan^2 x \, dx = \tan x + c$

$$\int \tan^2 x \, dx = \int (1 + \tan^2 x) - 1 \, dx$$

$$= \int 1 + \tan^2 x \, dx - \int 1 \, dx$$

$$= \underline{\tan x - x + C}$$

oppg.

$$\int \sin(3x+1)\,dx = \int \frac{1}{3}\cdot \underbrace{3}_{u'} \sin \underbrace{(3x+1)}_{u}\,dx$$

$$= \frac{1}{3}\left(-\cos(3x+1)\right) + C$$

$$= \underline{\frac{-1}{3}\cos(3x+1) + C}$$

$$\left(\begin{aligned} &(F(u(x)))' \\ &= F'(u(x)) \cdot u'(x) \\ &= f(u(x)) \cdot u'(x) \end{aligned} \right)$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$$= \int \frac{-u'}{u}\,dx$$

$$= -\int \frac{u'}{u}\,dx$$

$$= \underline{-\ln|\cos x| + C}$$

$u = \cos x$

$u' = -\sin x$

$$\left(\ln|u(x)|\right)' = \frac{u'}{u}$$

$$\int \sin^2 x \, dx$$

$$\cos(2x) = \underbrace{\cos^2 x}_{1-\sin^2 x} - \sin^2 x = 1 - 2\sin^2 x$$

så $\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right)$

$$\int \sin^2 x \, dx = \int \frac{1}{2}\left(1 - \cos(2x)\right) dx$$

$$= \frac{1}{2}\left[\int 1 \, dx - \int \cos 2x \, dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C$$

$$\int \sin^2 x = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

$$= \frac{x}{2} - \frac{\sin x \cdot \cos x}{2} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int \frac{1}{(2x-5)^2}\,dx = \int (2x-5)^{-2}\,dx$$

$$= \frac{1}{2}\cdot\frac{(2x-5)^{-1}}{-1} + c$$

$$= \frac{-1}{2(2x-5)} + C$$

e^{-x^2} har ingen antiderivert som er en "elementær" funksjon

Øving

$\int \tan x \, dx$

Deriver $\ln|\cos x|$

Bruker kjerneregelen: $\frac{1}{\cos x} \cdot \underbrace{(\cos x)'}_{-\sin x} = -\frac{\sin x}{\cos x}$

så $(-\ln|\cos x|)' = \tan x$

Derfor er $\int \tan x \, dx = \underline{-\ln|\cos x| + C}$

generelt $(\ln|u(x)|)' = \frac{u'(x)}{u(x)}$

så $\int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + C$

ex: $\int \frac{2x}{x^2+1} dx = \int \frac{u'(x)}{u(x)} dx = \underline{\ln|x^2+1| + C}$

$u(x) = x^2+1$

og $u'(x) = 2x$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

Fordi $(\arctan x)' = \frac{1}{1+x^2}$. vi viser dette:

$\tan(\arctan x) = x$ deriver

$(\tan)'(\arctan x) \cdot (\arctan x)' = 1$

$$\underbrace{\overbrace{(1+\tan^2)}^{(\tan)'}(\arctan x)}_{(1+x^2)}(\arctan x)' = 1$$

Så $(\arctan x)' = \frac{1}{1+x^2}$.

En gang til

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-u'}{u}\,dx = -\ln|u| + C$$
$$= -\ln|\cos x| + C$$

$(u = \cos x \;,\; \sin x = -u')$

$$\int -3 + 4x + e^{-2x}\, dx$$

$$= \int -3\,dx + 4\int x\,dx + \int e^{-2x}\,dx$$

$$= -3x + 4\frac{x^2}{2} + \frac{e^{-2x}}{-2} + c$$

$$= -3x + 2x^2 - \frac{1}{2}e^{-2x} + c$$

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{u'}{u}\,dx \underset{(u=1+e^x)}{} = \underline{\ln|1+e^x| + c}$$

LIKE!

$$\int \frac{1}{1+e^{-x}}\,dx = \int \frac{1}{1+e^{-x}} \cdot \frac{e^x}{e^x}\,dx = \int \frac{e^x}{e^x+1}\,dx$$

Altemativt: $\int \frac{1}{1+e^{-x}}\,dx$

$$= \ln|1+e^{-x}| + c.$$

$u = 1+e^{-x}$

$u' = -e^{-x}$

$$\frac{1}{1+e^{-x}} = \frac{1+e^{-x}-e^{-x}}{1+e^{-x}} = \frac{1+e^{-x}}{1+e^{-x}} - \frac{e^{-x}}{1+e^{-x}}$$

$$= 1 + \frac{-e^{-x}}{1+e^{-x}}$$

$$\int \frac{1}{1+e^{-x}}\,dx = \int 1 + \frac{-e^{-x}}{1+e^{-x}}\,dx$$

$$= x + \ln|1+e^{-x}| + c$$

Dette er lik $\ln|1+e^{x}| + c$

$$x = \ln(e^{x})$$

$$\underline{x + \ln(1+e^{-x})} = \ln(e^{x}) + \ln(1+e^{-x})$$

$$= \ln(e^{x} \cdot (1+e^{-x}))$$

$$= \underline{\ln(1+e^{x})}$$

$$\underbrace{(\ln(e^x))'}_{(x)'} = \frac{1}{e^x}\cdot(e^x)' = \frac{e^x}{e^x} = \underline{1}$$

$$(x)' = 1.$$

$$\int \frac{1}{4+x^2}\,dx \qquad \frac{1}{x^2+4} = \frac{1/4}{\frac{x^2}{4}+1} = \frac{1/4}{\left(\frac{x}{2}\right)^2+1}$$

$$= \frac{1}{4}\int \frac{2\cdot(1/2)}{(x/2)^2+1}\,dx = \frac{1}{4}\cdot 2\arctan\left(\frac{x}{2}\right)+c$$

$$= \underline{\frac{1}{2}\arctan\left(\frac{x}{2}\right)+c}$$

Minner om $\displaystyle\int f(ax+b)\,dx = \frac{F(ax+b)}{a}+c$

$$\int (3x-1)^4 \, dx = \int f(ax+b) \, dx$$

$f(u) = u^4$ og $a = 3$

$b = -1$

$F(u) = \frac{u^5}{5}$

$$\int (3x-1)^4 \, dx = \left(\frac{(3x-1)^5}{5}\right) \cdot \frac{1}{3} + C$$

$$= \frac{(3x-1)^5}{15} + C$$