

8mars
2022

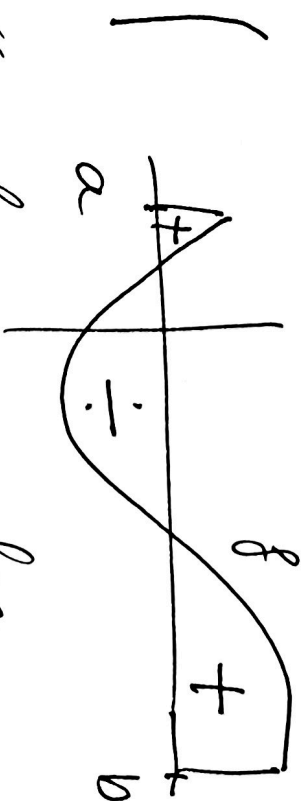
15.6 Fundamentalthæoremet i kalkulus

Ubæstemt integral

$$\int f(x) dx = F(x) + C \quad \text{hvor } F'(x) = f(x)$$

Beskriv integral

$$\int_a^b f(x) dx = \text{"areal med fortegn"}$$



Areal mellem $f(x)$ og x -aksen
+ arealer over x -aksen
- arealer under x -aksen

Fundamentalthæoremet

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b$$

for en antiderivat F af f .

Arla $f(x)$ er kontinuert
på $[a, b]$.

Dette er uafhængig af valg af antiderivat

La $G(x)$ være en anden antiderivat.

Da $G(x) = F(x) + c$ for en konstant c

$$\frac{G(b) - G(a) = (F(b) + c) - (F(a) + c)}{=} F(b) - F(a) + \underbrace{c - c}_0$$

$$y=x \quad \int x \, dx = \frac{x^2}{2} + c$$



Fund. lmm:

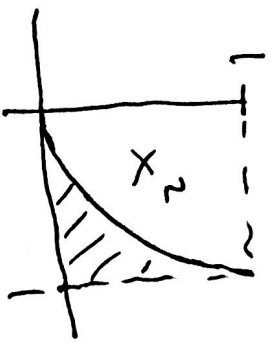
$$\int_0^1 x \, dx = F(1) - F(0), \text{ hvor } F(x) = \frac{x^2}{2}$$

$$= \frac{1}{2} - 0 = \frac{1}{2} \checkmark$$

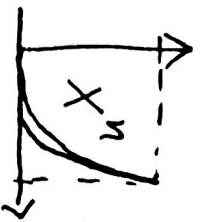
$$= G(1) - G(0) \quad \text{hvor } G(x) = \frac{x^2}{2} - 13$$

$$= \left(\frac{1}{2} - 13 \right) - \left(-13 \right) = \frac{1}{2} - 13 + 13 = \frac{1}{2} \checkmark$$

$$\left(\int_0^1 x \, dx = \frac{x^2}{2} \Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} \right)$$

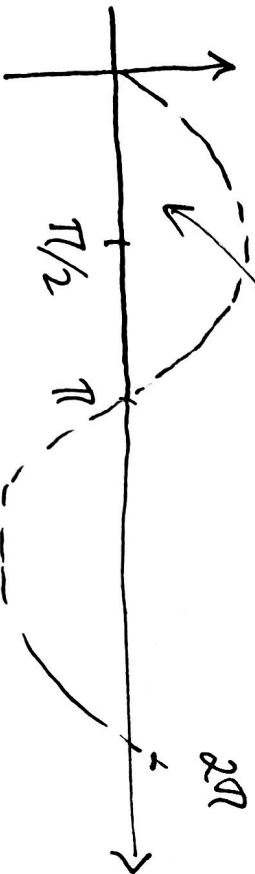


$$\int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \underline{\underline{\frac{1}{3}}}$$

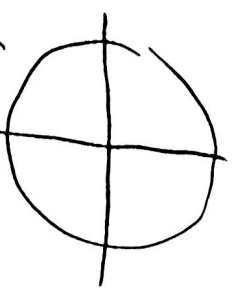


$$x^n \quad n \geq 1 \quad \int_0^1 x^n dx = \frac{x^{n+1}}{n+1} \Big|_0^1 = \underline{\underline{\frac{1}{n+1}}}$$

area like 2.



$$\begin{aligned} & \int_0^\pi \sin x \, dx \\ &= -\cos x \Big|_0^\pi \quad \leftarrow \text{antiderivativ til } \sin x \\ &= -\cos(\pi) - (-\cos(0)) \\ &= -(-1) + 1 = \underline{\underline{2}} \end{aligned}$$



$$(\cos x)' = -\sin x \quad \text{så} \quad (-\cos x)' = \sin x.$$

adding
 as variable
 for
 sin(2x).

$$\int \underbrace{2 \sin x \cos x}_{\sin(2x)} dx = \int \sin 2x dx = \frac{-\cos(2x)}{2} + C$$

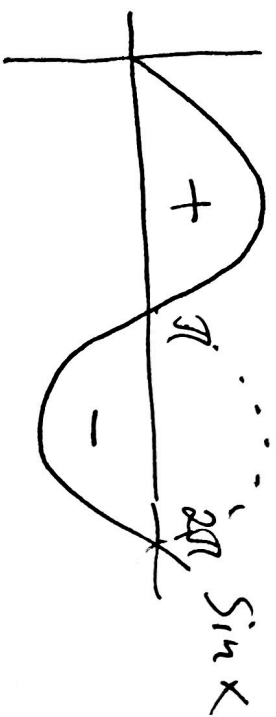
benytte lineær substitution $\int f(ax+b) dx = \frac{F(ax+b)}{a} + C$

$$\int_0^\pi \sin 2x dx = \left. \frac{-\cos 2x}{2} \right|_0^\pi = \frac{-1}{2} (\cos(2\pi) - \cos(0)) = \underline{\underline{0}}$$

Formlen følger fra addisjonsformelen: $\sin(x+y) = \sin x \cos y + \sin y \cos x$

La nå $x=y$: $\sin(x+x) = \sin(2x) = 2 \sin x \cdot \cos x$

Se 11.1 og 11.2 i boka.



$$\int_0^{2\pi} \sin x \, dx = 0$$

like start area under
x-axis, sum over.

How are areas between $\sin x$ and x-axis from $x=0$ to $x=2\pi$?

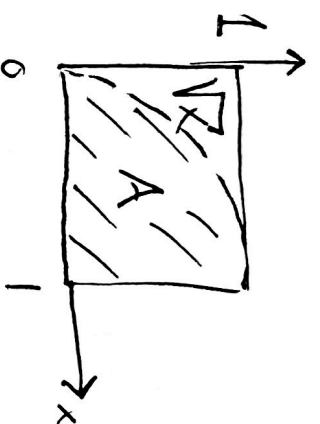
$$\int_0^{2\pi} |\sin x| \, dx = \int_0^{\pi} \sin x \, dx + \underbrace{\int_{\pi}^{2\pi} -\sin x \, dx}_{\text{like start}}$$

$$= 2 \int_0^{\pi} \sin x \, dx = 2 \cdot 2 = \underline{\underline{4}}$$

opgave

Find arealet A

$$A = 2/3$$



$$y = \sqrt{x}$$

$$y^2 = x$$

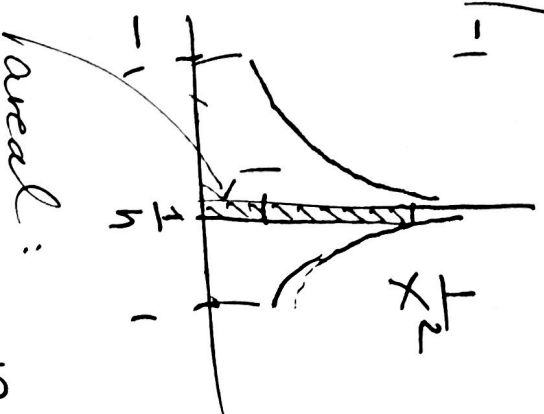
$$\begin{aligned} \int_0^1 \sqrt{x} \, dx &= \int_0^1 x^{1/2} \, dx = \frac{x^{1/2+1}}{3/2} \Big|_0^1 \\ &= \frac{1}{3/2} \cdot \frac{2}{2} = \frac{2}{3} \end{aligned}$$

Ellipsen i kjet.

$$\begin{aligned} \int_{-1}^1 \frac{1}{x^2} \, dx &= \int_{-1}^1 x^{-2} \, dx = \frac{x^{-1}}{-1} \Big|_{-1}^1 = -\frac{1}{x} \Big|_{-1}^1 \\ &= -1 - \left(-\frac{1}{-1}\right) = -2 \end{aligned}$$

$\frac{1}{x}$ er ikke en antiderivat til $\frac{1}{x^2}$ på $[-1, 1]$

Feiler p. nar $x=0$.



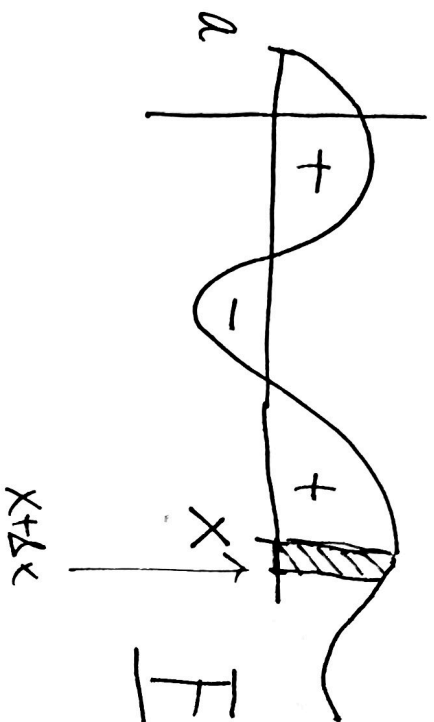
areal:

$$\frac{1}{n} \cdot \left(\frac{1}{n}\right)^2 = \frac{1}{n^3}$$

bredde · høyde

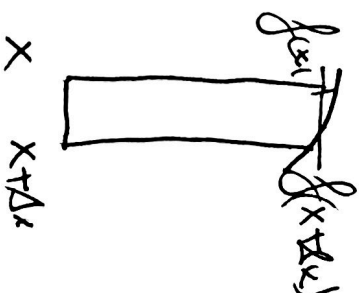
Bevis for fundamentalteoremet.

$$F(x) = \int_a^x f(x) dx$$



$$\frac{F(x+\Delta x) - F(x)}{\Delta x}$$

= gennemsnitts
højde til



$$\lim_{\Delta x \rightarrow 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x)$$

hvis f er kontinuert

$F(x)$ er en antiderivat til $f(x)$.

Hvis $G(x)$ er en anden antiderivat

$$F(x) = G(x) + C$$

så $C = -G(a)$

$x=a$

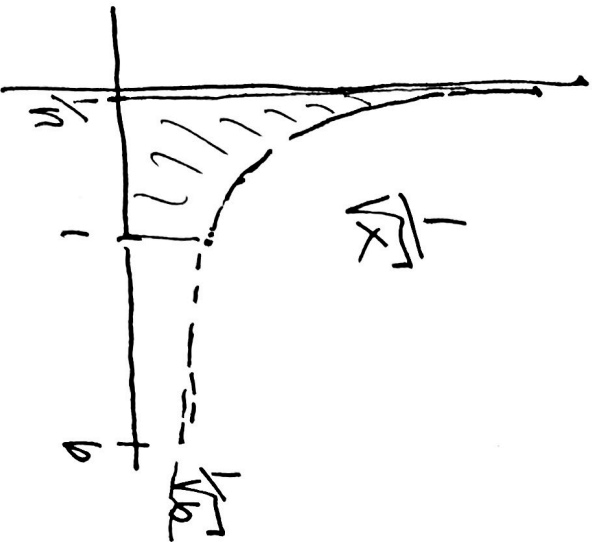
$$F(a) = \int_a^a f(x) dx = 0 = G(a) + C$$

$$F(x) = \int_a^x f(x) dx = G(x) - G(a)$$

$$\int_a^b f(x) dx = G(b) - G(a) = G(x) \Big|_a^b$$

L_a x = b
L_b x = a

h_{va} G er antiderivat
hil f(x).

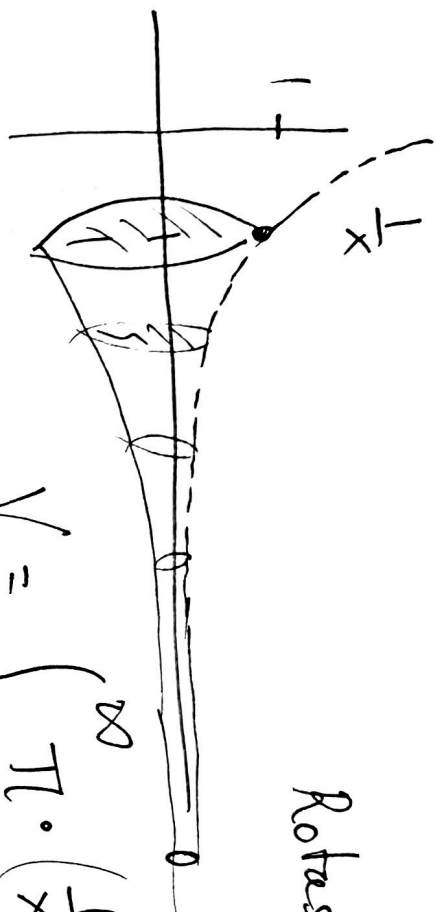


$$\int_{1/n}^1 \frac{1}{\sqrt{x}} dx = \int_{1/n}^1 x^{-1/2} dx = \frac{x^{1/2}}{1/2} \Big|_{1/n}^1 = 2x^{1/2} \Big|_{1/n}^1$$

$$= \frac{2 - 2\sqrt{1/n}}{1}$$

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{n \rightarrow \infty} \int_{1/n}^1 \frac{1}{\sqrt{x}} dx = 2$$

Rotationslegeme



Gabriel's
Horn

$$V = \int_1^{\infty} \pi \cdot \left(\frac{1}{x}\right)^2 dx$$

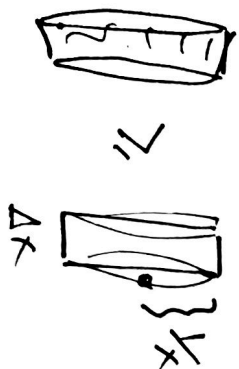
$$= \lim_{N \rightarrow \infty} \int_1^N \pi \left(\frac{1}{x}\right)^2 dx$$

$$\pi \left[-\frac{1}{x} \right]_1^N$$

$$\lim_{N \rightarrow \infty}$$

$$\pi \left[-\frac{1}{x} \right]_1^N$$

$$= \lim_{N \rightarrow \infty} \pi \left(1 - \frac{1}{N} \right) = \pi$$

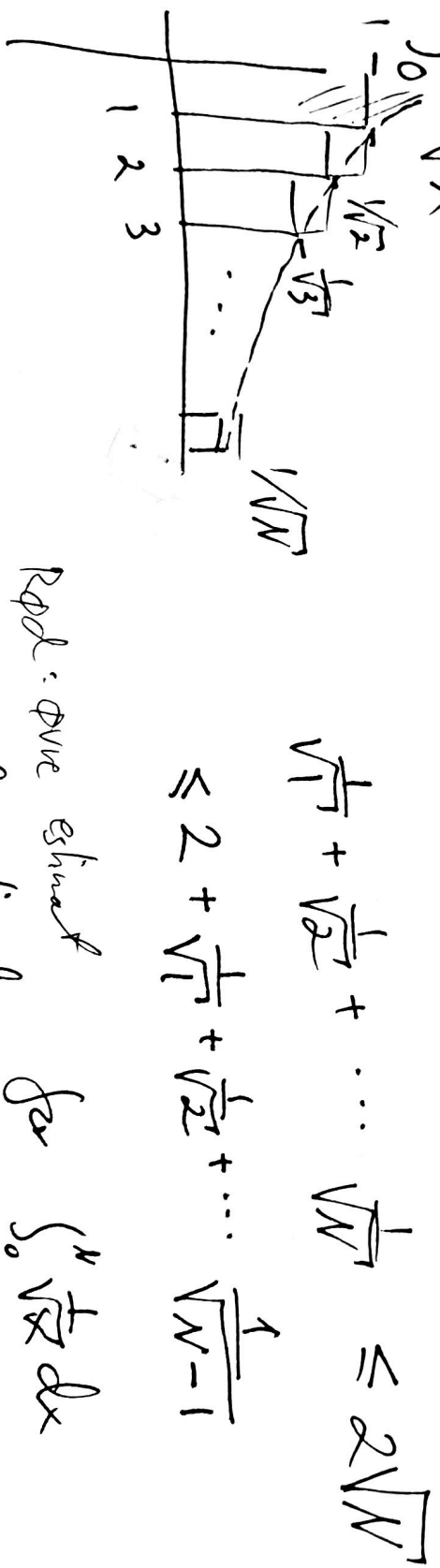


overflacheneal $> \int_1^{\infty} 2\pi \cdot \frac{1}{x} dx$

$$\begin{aligned}
 & \lim_{N \rightarrow \infty} \int_1^N 2\pi \cdot x^{-1} dx \\
 &= \lim_{N \rightarrow \infty} 2\pi \ln(x) \Big|_1^N = \lim_{N \rightarrow \infty} 2\pi (\ln(N) - \ln(1)) \\
 &= \infty
 \end{aligned}$$

So Gabriel's Horn has vanishing surface area.

$$\int_0^N \frac{1}{\sqrt{x}} dx = \int_0^N x^{-1/2} dx = 2 \cdot x^{1/2} \Big|_0^N = 2\sqrt{N}$$



$$\begin{aligned}
 & \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{N}} \leq 2\sqrt{N} \\
 & \leq 2 + \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{N-1}}
 \end{aligned}$$

Repd: One estimate for $\int_0^N \frac{1}{\sqrt{x}} dx$
 sort: under estimate

$$15.66 \quad f(x) = \frac{2}{x}$$

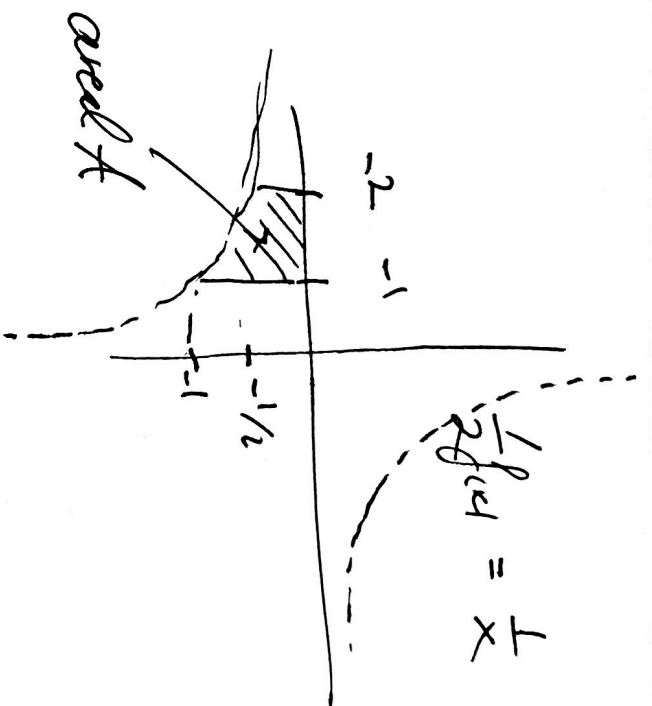
$$A = - \int_{-2}^{-1} f(x) dx$$

$$= -2 \int_{-2}^{-1} \frac{1}{x} dx$$

$$= -2 \ln|x| \Big|_{-2}^{-1}$$

$$= -2 (\ln|-1| - \ln|-2|)$$

$$= 2 \ln 2 \sim 1.386 \dots$$

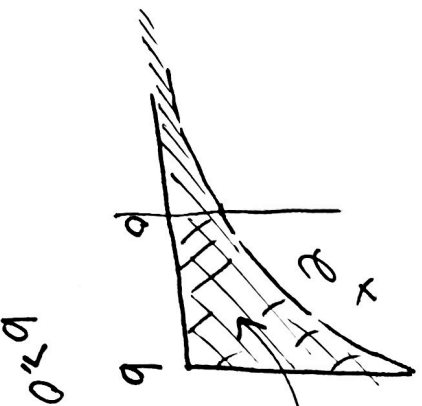


Hva er arealet? $\int_0^b e^x dx = e^x \Big|_0^b = \underline{e^b - 1}$

Hva er arealet til det røde område?

$$\int_{-\infty}^b e^x dx = \lim_{N \rightarrow \infty} \int_{-N}^b e^x dx = \lim_{N \rightarrow \infty} (e^b - e^{-N})$$

$$= \underline{e^b}$$



$$f(x) = x^2 - 4 \quad \text{1) Region 1} \quad \int_0^4 f(x) dx$$

opposite

2) Region 2 area is measured as $f(x)$, x-axis.

$$x = 0 \text{ to } x = 4$$

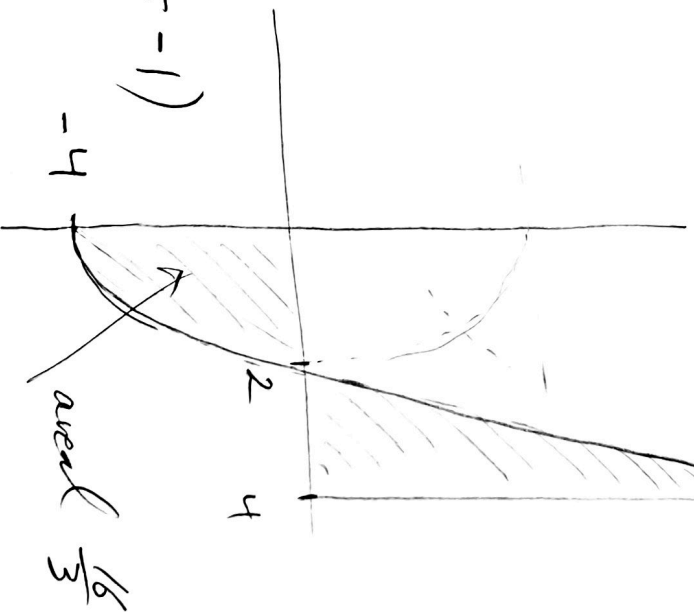
$$A = -\int_0^2 f(x) dx + \int_2^4 f(x) dx$$

$$= \int_0^2 |f(x)| dx$$

$$\int f(x) dx = \int x^2 - 4 dx = \frac{x^3}{3} - 4x + C$$

$$\int_0^2 f(x) dx = \left. \frac{x^3}{3} - 4x \right|_0^2 = \frac{8}{3} - 8 = 8\left(\frac{1}{3} - 1\right)$$

$$= -\frac{16}{3}$$



$$\int_2^4 f(x) dx = \left. \frac{x^3}{3} - 4x \right|_2^4 = \left(\frac{4^3}{3} - 4 \cdot 4 \right) - \left(\frac{2^3}{3} - 8 \right)$$

$$= \frac{1}{3} [4^3 - 4^2 \cdot 3 + 16] = \frac{1}{3} [4^2(4-3) + 16] = \frac{32}{3}$$

$$\begin{aligned}
 \int_0^4 f(x) dx &= \int_0^2 f(x) dx + \int_2^4 f(x) dx = -\frac{16}{3} + \frac{2 \cdot 16}{3} = \frac{(-1+2) \cdot 16}{3} = \underline{\underline{\frac{16}{3}}} \\
 \text{Area} &= -\int_0^2 f(x) dx + \int_2^4 f(x) dx = -\left(-\frac{16}{3}\right) + \frac{2 \cdot 16}{3} = \frac{3 \cdot 16}{3} = \underline{\underline{16}} \\
 \text{Riemann sum} \int_0^4 f(x) dx &= \frac{x^3}{3} - 4x \Big|_0^4 = \frac{4^3}{3} - 4 \cdot 2 = \frac{4^2}{3} = \underline{\underline{\frac{16}{3}}}
 \end{aligned}$$

Example

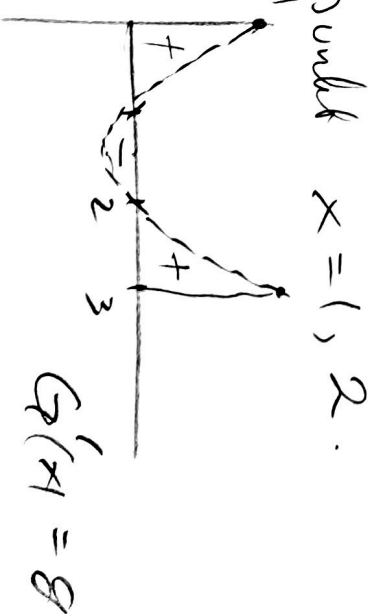
$$\int_0^3 \underbrace{x^2 - 3x + 2}_{g(x)} dx$$

Find area between $g(x)$ and x -axis for $x=0$ to $x=3$.

$$\begin{aligned}
 g(x) &= x^2 - 3x + 2 = (x-2)(x-1) \\
 g(0) &= 2 \quad g(3) = 9 - 3^2 + 2 = 2
 \end{aligned}$$

$$\text{Area} A = \int_0^1 g(x) dx - \int_1^2 g(x) dx + \int_2^3 g(x) dx$$

$$(\underline{G(1)} - G(0)) - (G(2) - \underline{G(1)}) + (G(3) - G(2))$$



$$A = 2G(1) - G(1) - 2G(2) + G(3)$$

$$\int x^2 - 3x + 2 \, dx = \frac{x^3}{3} - 3\frac{x^2}{2} + 2x = G(x)$$

$$A = 2 \left(\frac{1}{3} - \frac{3}{2} + 2 \right) - 0 - 2 \left(\frac{8}{3} - 3 \cdot 2 + 4 \right) + \left(9 - \frac{27}{2} + 6 \right)$$

$$= \frac{2}{3} - 3 + 4 + 4 - \frac{16}{3} + 15 - \frac{27}{2}$$

$$= \frac{2}{3} + 5 - \frac{16}{3} + 15 - 13.5$$

$$= \frac{2}{3} + 5 - \frac{16}{3} + 15 - 13.5$$

$$= \frac{2}{3} + 5 - \frac{16}{3} + 15 - 13.5$$

$$= \frac{2}{3} + 5 - \frac{16}{3} + 15 - 13.5$$

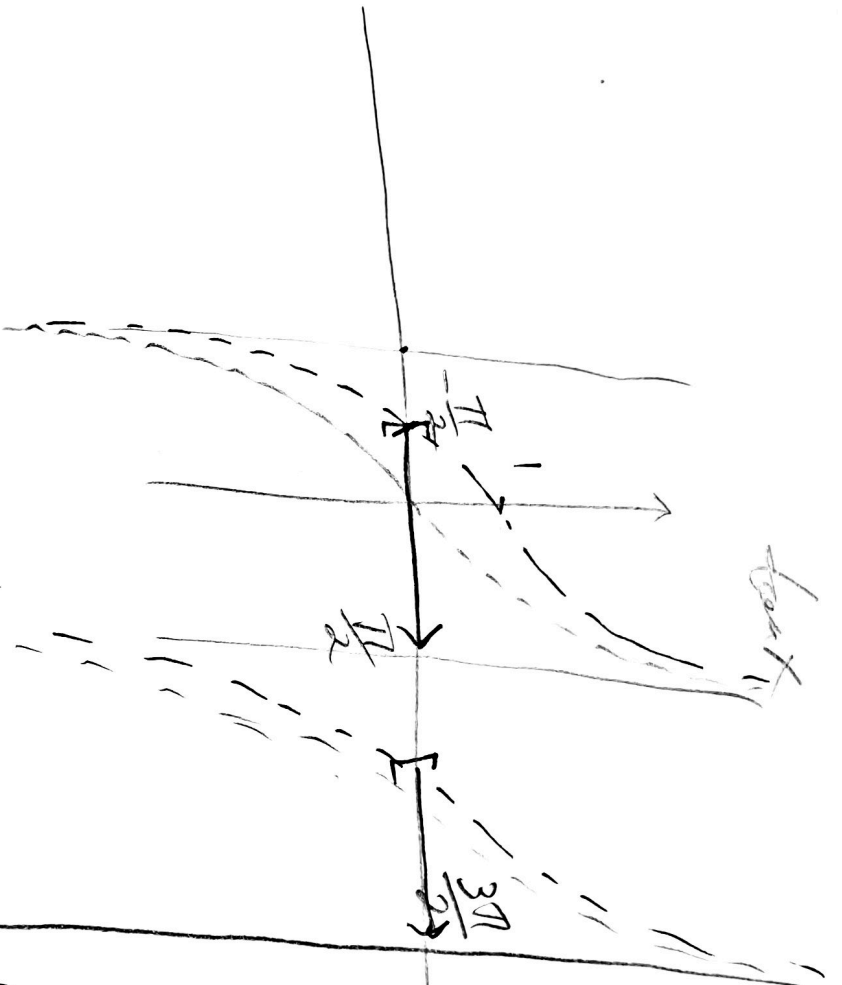
$$= \frac{2}{3} - \frac{4}{3} + 6.5$$

$$= \frac{15-4}{6}$$

$$= \frac{11}{6}$$

11/6

Übung 6 #10



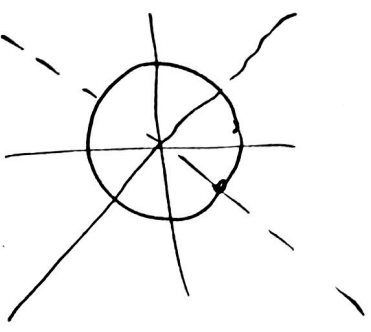
vertikale asymptoten
 $x = \frac{\pi}{2} + \pi \cdot n$
 $n \in \mathbb{Z}$

$$\tan x + 1$$

Nulldpunkt:

$$\tan x = -1$$

$$x = -\frac{\pi}{4} + \pi \cdot n$$



$$\tan x + 1 \geq 0$$

$$\left[-\frac{\pi}{4}, \frac{\pi}{2} \right)$$

offene Periode π

$$\bigcup_{n \in \mathbb{Z}} \left[-\frac{\pi}{4} + \pi \cdot n, \frac{\pi}{2} + \pi \cdot n \right)$$