10 mars 2021

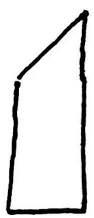
15.9

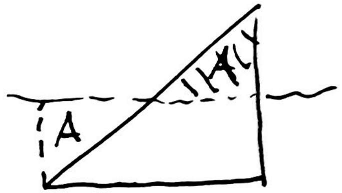
Numerisk integrasjon

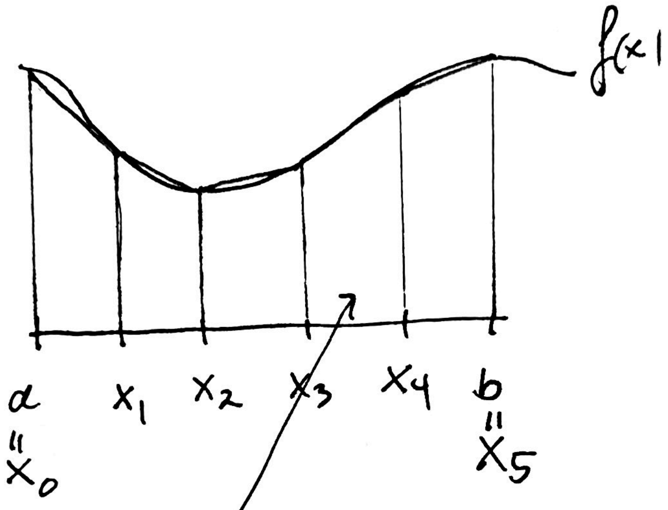
Vi illustrerte trapesmetoden ved bruk av geogebra. :

Trapezoidalsum(...)

Trapes metoden







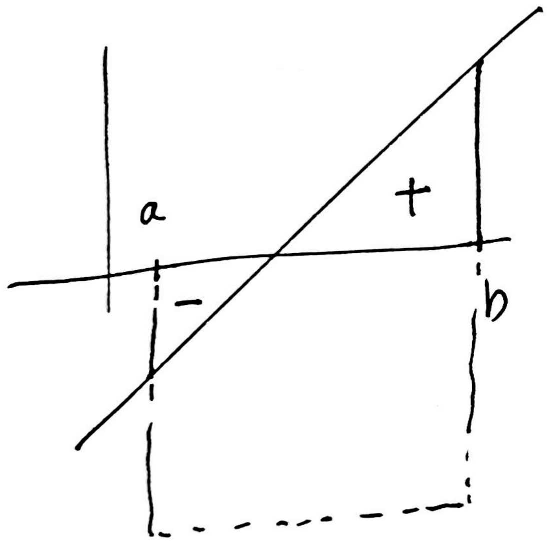
indelt i $n=5$ delintervaller

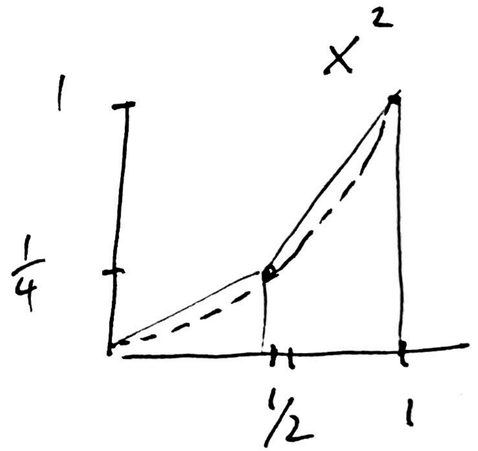
arealet til trapeset er $\left(\frac{f(x_3)+f(x_4)}{2}\right)(x_4-x_2)$

Bruker gjerne delintervaller med lik lengde. $\frac{b-a}{n}$

$$\int_a^b f(x)\,dx \overset{\text{trapes}}{\sim} \frac{b-a}{n}\left[\frac{f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)}{2}\right]$$

Hvis $f(x) = kx + l$, lineær funksjon, da gir trapesmetoden integralet eksakt (for alle $n \geq 1$)





velger 2 delintervaller

Eksempel $f(x) = x^2$

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}. \text{ eksakt}$$

$$\underset{\text{trapes, } n=2}{T_2} = \underset{\text{bredde}}{\frac{1}{2}} \left[\frac{f(0) + f(\frac{1}{2})}{2} + \frac{f(\frac{1}{2}) + f(1)}{2} \right] = \frac{1}{2} \cdot \frac{1}{2} \left[f(0) + 2f(\tfrac{1}{2}) + f(1) \right]$$

$$= \frac{1}{4}\left[0 + 2 \cdot \frac{1}{4} + 1\right]$$

$$= \frac{1}{4} \cdot \frac{3}{2} = \frac{3}{8} = \underline{0.375}$$

oppg Finn $T_4 = \frac{1}{4}\left[\frac{f(0)+f(\frac{1}{4})}{2} + \frac{f(\frac{1}{4})+f(\frac{1}{2})}{2} + \frac{f(\frac{1}{2})+f(\frac{3}{4})}{2} + \frac{f(\frac{3}{4})+f(1)}{2}\right]$

$n=4$

$\frac{b-a}{n} = \frac{1}{4}$ bredde

$$T_4 = \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{4} + \frac{1}{2}\cdot 1\right]$$

$$= \frac{1}{4}\cdot\frac{1}{16}\left[1 + 4 + 9 + 8\right]$$

$$= \frac{22}{4\cdot 16} = \frac{11}{32} \sim \underline{0.343}$$

(eksakt: 0.333...)

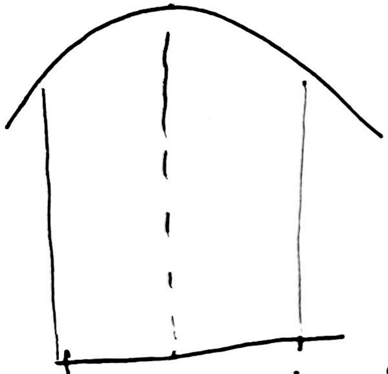
x	x^2
0	0
1/4	1/16
1/2	1/4
3/4	9/46
1	1

Nøyaktighet til trapesmetoden:

Anta $f(x)$ er 2 ganger deriverbar og $|f^{(2)}(x)| \leq M_2$ i $[a,b]$

$$\left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{M_2\,(b-a)^3}{12\cdot n^2}$$

Simpsons metode



$\frac{1}{6}$ $\frac{4}{6}$ $\frac{1}{6}$

Denne vektingen gir eksakt resultat for alle kvadratiske uttrykk

Vekting:

$$\left.\begin{array}{ccccccccc} 1 & 4 & 1 & & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & & 1 & 4 & 1 & & \\ & & & & & & 1 & 4 & 1 \end{array}\right\} \text{4 doble delintervaller}$$

$$\begin{array}{ccccccccc} \hline 1 & 4 & 2 & 4 & 2 & 4 & 2 & 4 & 1 \end{array}$$

Deler intervallet $[a,b]$ i $2n$ biter

$$\frac{b-a}{n}\left[\frac{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots 4f(x_{2n-1}) + f(x_{2n})}{6}\right]$$

$f(x) = x^3.$

$$\int_0^1 x^3 dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$S_1 = \frac{1-0}{1}\left[\frac{f(0) + 4\cdot f(\frac{1}{2}) + f(1)}{6}\right]$$

$$= \frac{1}{6}\left(0 + 4\left(\frac{1}{2}\right)^3 + 1\right) = \frac{1}{6}\left(\frac{1}{2}+1\right) = \frac{1}{6}\cdot\frac{3}{2}$$

$$= \underline{\frac{1}{4}}$$

gir eksakt resultat.

$(T_2$ gir $0.281\ldots)$

$$\int_0^1 x^4 dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5} = 0.2$$

$$S_1 = \frac{1}{1}\cdot\frac{1}{6}\left[0 + 4\cdot\left(\frac{1}{2}\right)^4 + 1^4\right] = \frac{1}{6}\left[\frac{1}{4}+1\right] = \frac{1}{6}\cdot\frac{5}{4} = \frac{5}{24} \sim 0.2083\ldots$$

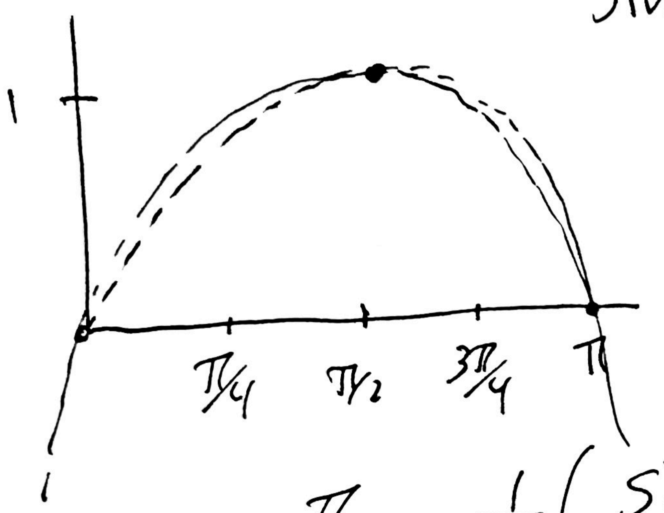
$$S_2 = \frac{1}{2}\cdot\frac{1}{6}\left[0 + 4\cdot\left(\frac{1}{4}\right)^4 + 2\cdot\left(\frac{1}{2}\right)^4 + 4\cdot\left(\frac{3}{4}\right)^4 + 1^4\right] = \frac{1}{2}\cdot\frac{1}{6}\left[\frac{1}{4^3} + \frac{1}{8} + \frac{81}{4^3} + 1\right]$$

$$= \frac{1}{2}\frac{1}{6\cdot 4^3}\left[1 + 8 + 81 + 64\right]$$

$$= \frac{154}{2\cdot 3\cdot 128} = \frac{154}{3\cdot 256} = \frac{77}{384}$$

$$\underline{\underline{\sim 0.20052}}$$

Sin x



$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi}$$
$$= -\cos(\pi) - (-\cos(0))$$
$$= \underline{2}$$

$$S_1 = \frac{\pi}{1} \cdot \frac{1}{6}\Big(\underbrace{\sin(0)}_{0} + \underbrace{4\sin\left(\tfrac{\pi}{2}\right)}_{4} + \underbrace{\sin \pi}_{0} \Big) = \frac{\pi \cdot 4}{6} = \frac{2}{3}\pi$$
$$\sim 2.0944$$

$$S_2 = \frac{\pi}{2} \cdot \frac{1}{6}\Big(\sin 0 + 4\sin\left(\tfrac{\pi}{4}\right) + 2\sin\left(\tfrac{\pi}{2}\right) + 4\sin\left(\tfrac{3\pi}{4}\right) + \sin \pi \Big)$$
$$\left(4 \cdot \tfrac{1}{\sqrt{2}} + 2 + 4 \cdot \tfrac{1}{\sqrt{2}} + 0 \right)$$
$$= \frac{\pi}{12}\left(0 + 4 \cdot \tfrac{1}{\sqrt{2}} + 2 + 4 \cdot \tfrac{1}{\sqrt{2}} + 0 \right)$$
$$= \frac{\pi}{12}\Big(\big(\underbrace{2 \cdot \tfrac{1}{\sqrt{2}}}_{\sqrt{2}}\big) \cdot 4 + 2 \Big) = \frac{2\pi}{12}\left(2\sqrt{2} + 1 \right) = \frac{\pi(2\sqrt{2}+1)}{6}$$
$$\sim \underline{\underline{2.00456}}$$

Feilestimat Simsons metode

$$|f^{(4)}_{(x)}| < M_4 \quad i \quad [a,b]$$

$$\left| \int_a^b f(x)\,dx - S_n \right| \leq \frac{M_4 \, (b-a)^5}{180 \cdot n^4}$$

$f(x)$ konst $f(x) = c$ Simpsons gir eksakt resultat

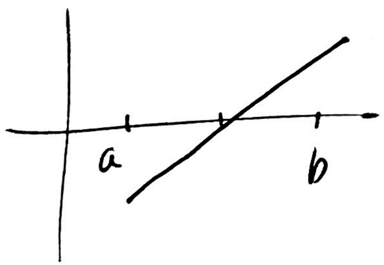
$$\frac{b-a}{1}\left(\frac{c}{6} + \frac{4c}{6} + \frac{c}{6} \right) = (b-a)\cdot c \checkmark$$

$f(x) = k \cdot x$

$$\frac{b-a}{1}\left(\frac{ka}{6} + \frac{4}{6} k\left(\frac{b+a}{2}\right) + \frac{kb}{6} \right)$$

$$= k \cdot \frac{b-a}{6}\left(a + 2(b+a) + b \right) = \frac{(b-a)}{6} \cdot (b+a) \cdot 3k$$

$$= \frac{(b-a)(b+a)}{2} k = \frac{b^2 - a^2}{2} k \checkmark$$



sjekker at Simpsons metode er eksakt for $f(x) = x^2$.

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{1}{3}(b^3 - a^3)$$

$$= \frac{1}{3}(b-a)(b^2 + ab + a^2) \quad \text{eksakt}$$

$$S_1 = \frac{(b-a)}{1 \cdot 6}\left[a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right]$$

$$= \frac{b-a}{6}\left[a^2 + (a+b)^2 + b^2\right] = \frac{b-a}{6}\left[a^2 + a^2 + 2ab + b^2 + b^2\right]$$

$$= \frac{(b-a)(a^2 + ab + b^2)}{3} \quad \checkmark$$

Tilsvarende for x^3 (tidligere eks) ...

Simpsons metode gir eksakt resultat (for alle $n \geq 1$) for alle polynomer av grad ≤ 3.