22 mars

16.5 Delbrøkesoppspalting

Rasjonal funksjon $\frac{p(x)}{q(x)}$ p, q polynomer

$\frac{x+2}{x^2-1}$

Polynom divisjon $\frac{p(x)}{q(x)} = S(x) + \frac{r(x)}{q(x)}$ hvor $\deg r < \deg q$.

graden til r er ikke mindre enn graden til q.

Eksempel $\frac{x^2}{x-2} = x+2 + \frac{4}{x-2}$

$$
\begin{array}{l}
x^2+0\cdot x+0 : x-2 = x+2+\frac{4}{x-2} \\
\underline{x^2-2x} \\
\quad 2x+0 \\
\quad \underline{2x-4} \\
\qquad 4
\end{array}
$$

$$\int \frac{x^2}{x-2}\,dx = \int x+2 + \frac{4}{x-2}\,dx$$
$$= \int x+2\,dx + 4\int \frac{1}{x-2}\,dx$$
$$= \underline{\frac{x^2}{2} + 2x + 4\ln|x-2| + C}$$

(lineær subst

$\int \frac{1}{x-2}\,dx$ $u = x-2$, $u' = 1$, $dx = du$

$= \int \frac{1}{u}\,du = \ln|u| + C = \underline{\ln|x-2| + C}$)

$$\int \frac{1}{x^2-2x}\,dx \;?$$

$$\frac{1}{x^2-2x} = \frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

Bestemmer A og B:

Felles nevner gir $\dfrac{1}{x(x-2)} = \dfrac{A(x-2) + Bx}{x(x-2)}$

$$\Leftrightarrow \quad 1 = A(x-2) + Bx$$

$$= A\cdot x - 2A + Bx$$

$$0\cdot x + 1 = (A+B)x - 2A$$

$$\Leftrightarrow \quad A+B=0 \quad \text{og} \quad -2A = 1$$

Så $\underline{A = \frac{-1}{2}}$ og $B = -A = \underline{\frac{1}{2}}$

$$\frac{1}{x^2-2x} = \frac{1}{2}\left[\frac{-1}{x} + \frac{1}{x-2}\right] \checkmark$$ eksempel på delbrøksoppspalting.

$$\int \frac{1}{x^2-2x}\,dx = \frac{1}{2}\int \frac{-1}{x} + \frac{1}{x-2}\,dx = \frac{1}{2}\left[-\ln|x| + \ln|x-2|\right] + c$$

$$= \underline{\frac{1}{2}\ln\left|\frac{x-2}{x}\right| + c} = \left(\ln\sqrt{\frac{x-2}{x}} + c\right)$$

2

2

$$\int \frac{x-1}{x^2+3x+2}\,dx$$

(substitusjon
$u = x^2+3x+2$
$u' = 2x+3$ går ikke...)

$$x^2+3x+2 = (x+2)(x+1)$$

Delbrøksoppspalting: $\frac{x-1}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1}$

Finner A og B: Felles nevner gir $\frac{x-1}{(x+2)(x+1)} = \frac{A(x+1)+B(x+2)}{(x+2)(x+1)}$

$$x-1 = A(x+1) + B(x+2)$$

for alle x.

$x=-1$: $-1-1 = A\cdot 0 + B(\cancel{-1}+2)$

$\underline{-2 = B}$

$x=-2$: $-2-1 = A(-2+1) + B\cdot 0$

$-3 = -A$ så $\underline{A=3}$

($x-1 = (A+B)x + (A+2B)$
$\Updownarrow$
$A+B=1$ og $A+2B=-1$...)

2
2

$$\frac{x-1}{x^2+3x+2} = \frac{3}{x+2} - \frac{2}{x+1}$$

(sjekker: $\frac{3(x+1)-2(x+2)}{\sim} = \frac{x-1}{\sim}$ ✓)

$$\int \frac{x-1}{x^2+3x+2}\,dx = \int \frac{3}{x+2} - \frac{2}{x+1}\,dx$$

$$= \underline{3 \ln|x+2| - 2\ln|x+1| + C}$$

$$\left(= \ln\left|\frac{(x+2)^3}{(x+1)^2}\right| + C \right)$$

oppgave $\int \frac{x}{(2x+1)(x+1)}\,dx$

$$\frac{x}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

$$x = A(x+1) + B(2x+1) \text{ for alle } x.$$

$x=-1$: $\quad -1 = B(-2+1) = -B$: $\quad B = 1$

$x=-1/2$: $\quad -\frac{1}{2} = A\left(1-\frac{1}{2}\right) = \frac{1}{2}\cdot A$ så $\quad A = -1$

$$\frac{x}{(2x+1)(x+1)} = \frac{-1}{2x+1} + \frac{1}{x+1}$$ delbrøksoppspalting

$$\int \frac{x}{(2x+1)(x+1)}\,dx = -\int \frac{1}{2x+1}\,dx + \int \frac{1}{x+1}\,dx$$

$u = 2x+1$

$u' = 2$

$du = 2 \cdot dx$

så $\int \frac{1}{2x+1}\,dx = \int \frac{1}{u} \cdot \frac{1}{2}\,du = \frac{1}{2}\ln|u| + c$

$$= -\frac{1}{2}\ln|2x+1| + \ln|x+1| + c$$

$$\left(= \ln\left(\frac{|x+1|}{\sqrt{|2x+1|}}\right) + c \right)$$

$$\int \frac{1}{(x^2+1)(x-2)}\,dx$$

polynom av grad $1 < \deg(x^2+1) = 2$.

$$\frac{1}{(x^2+1)(x-2)} = \frac{Bx+C}{x^2+1} + \frac{A}{x-2}$$

felles nevner :

$$1 = (Bx+C)(x-2) + A(x^2+1)$$

$$= Bx^2 + Cx - 2Bx - 2C + Ax^2 + A$$

$$0 \cdot x^2 + 0 \cdot x + 1 = (A+B)x^2 + (C-2B)x + A - 2C$$

$A+B=0$ $\quad C-2B=0$ $\quad A-2C=1$

setter inn $x=2$: $\quad 1 = 0 + A(2^2+1)$ så $\underline{A = \frac{1}{5}}$

$B = -A = \frac{-1}{5}$. $\quad C = 2B = 2\left(\frac{-1}{5}\right) = \frac{-2}{5}$.

(Dette stemmer med $A - 2C = 1$

$\frac{1}{5} - 2\left(\frac{-2}{5}\right) = 1$ ✓)

$$\int \frac{1}{(x^2+1)(x+1)}\,dx = \int \frac{1/5}{x-2} + \frac{(-1/5)x - 2/5}{x^2+1}\,dx$$

$$= \frac{1}{5}\ln|x-2| + \frac{-1}{5}\int \frac{x}{x^2+1} + \frac{2}{x^2+1}\,dx$$

$$(\arctan x)' = (\tan^{-1} x)' = \frac{1}{1+x^2}$$

$$\int \frac{x}{x^2+1}\,dx \qquad u = x^2+1, \quad u' = 2x \quad \text{så} \quad \frac{1}{2}u'$$

$$= \int \frac{\frac{1}{2}u'}{u}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$$

$$\int \frac{1}{(x^2+1)(x+1)}\,dx = \underline{\frac{1}{5}\ln|x-2| + \frac{-1}{5}\cdot\frac{1}{2}\ln|x^2+1| - \frac{2}{5}\arctan(x) + C}$$

$$\int \frac{1}{(x+1)^2(x-1)}\,dx$$

$$\frac{1}{(x+1)^2(x-1)} = \frac{Ax+B}{(x+1)^2} + \frac{C}{x-1} \quad \ldots$$

$$= \frac{A(x+1) - A + B}{(x+1)^2} + \frac{C}{x-1}$$

$$= \frac{(B-A)}{(x+1)^2} + \frac{A}{(x+1)} + \frac{C}{x-1}$$

på formen $\dfrac{C_1}{x-1} + \dfrac{C_2}{(x+1)} + \dfrac{C_3}{(x+1)^2}$

$$\frac{1}{(x+1)^2(x-1)} = \frac{C_1}{x-1} + \frac{C_2}{(x+1)} + \frac{C_3}{(x+1)^2}$$ felles nevner :

$$1 = C_1(x+1)^2 + C_2(x+1)(x-1) + C_3(x-1)$$

$x=1$: $\quad 1 = c_1 \cdot 2^2 + 0 + 0$ $\quad$ gir $c_1 = \frac{1}{4}$.

$x=-1$: $\quad 1 = -2c_3$ $\quad$ gir $c_3 = \frac{-1}{2}$.

Deriverer begge sider m.h.t x $\quad 0 = 2c_1(x+1) + c_2(x-1) + c_2(x+1) + c_3$

setter $x=-1$ $\quad 0 = c_2(-2) + c_3$

$$c_2 = \frac{1}{2} c_3 = -\frac{1}{4}.$$

$$\int \frac{1}{(x+1)^2(x-1)}\,dx = \int \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{(x+1)^2}\,dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{-1}{2}\int \frac{1}{(x+1)^2}\,dx$$

$u = x+1$
$u' = 1$

$$\left(\int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C \right)$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\frac{1}{x+1} + C$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\frac{1}{x+1} + C$$

Oppgave: $\displaystyle\int \frac{x^2}{x^2+4x+4}\,dx$

1) Polynomdivisjon
2) Delbrøksoppspalting.

$$\frac{x^2+4x+4-4x-4}{x^2+4x+4} = 1 - 4\,\frac{x+1}{x^2+4x+4} = 1 - 4\,\frac{x+1}{(x+2)^2}$$

$$\frac{x+1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$$

$$x+1 = Ax + B+2A$$

Så $A = 1$, $B = -1$

$$\int \frac{x^2}{x^2+4x+4}\,dx = \int 1 - 4\left(\frac{1}{x+2} + \frac{-1}{(x+2)^2}\right) dx$$

$$= x - 4\ln|x+2| + 4\,\frac{-1}{x+2} + C$$

$$= \underline{x - 4\ln|x+2| - \frac{4}{x+2} + C}$$

$$\int \frac{x^3}{x^2+4}\,dx$$

$$\frac{(x^3+4x)-4x}{x^2+4} = x - \frac{4x}{x^2+4}$$

$u = x^2+4$

$u' = 2x$

$\frac{1}{2}du = x\,dx$

$$\int \frac{x^3}{x^2+4}\,dx = \int x\,dx - 4\underbrace{\int \frac{x}{x^2+4}\,dx}_{\int \frac{1}{u}\cdot\frac{1}{2}du}$$

$$= \frac{x^2}{2} - 4\cdot\frac{1}{2}\ln|x^2+4| + c$$

$$= \frac{x^2}{2} - 2\ln(x^2+4) + c$$

$$\int \frac{1}{x^4-1}\,dx$$

$$x^4-1 = (x^2)^2-1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$$

$$\frac{1}{x^4-1} = -\frac{1/2}{x^2+1} + \underbrace{\frac{1/2}{x^2-1}}_{-\frac{1/4}{x+1}+\frac{1/4}{x-1}}$$

$$= -\frac{1}{2}\cdot\frac{1}{x^2+1} - \frac{1}{4}\cdot\frac{1}{x+1} + \frac{1}{4}\cdot\frac{1}{x-1}$$

Systematisk: $\frac{1}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$ Felles nevner:

$$1 = (Ax+B)(x^2-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1)$$

for alle x.

$x = 1$: $\quad 1 = D \cdot 2 \cdot 2 \qquad \underline{D = \frac{1}{4}}$

$x = -1$: $\quad 1 = ((-1)^2+1)(-1-1) \qquad \underline{C = \frac{-1}{4}}$

setter inn for C, D:

$$1 = (Ax+B)(x^2-1) + \underbrace{\frac{-1}{4}(x^2+1)(x-1) + \frac{1}{4}(x^2+1)(x+1)}_{\frac{1}{4}(x^2+1)\left(\underbrace{-(x-1) + (x+1)}_{2}\right)}$$

$$1 = A \cdot x^3 + Bx^2 - Ax - B + \frac{1}{2}(x^2+1) \qquad \text{så } \underline{A = 0}$$

$$1 = Bx^2 + \frac{1}{2}x^2 + -B + \frac{1}{2} \qquad \text{så } \underline{B = -1/2}$$

$$\int \frac{1}{x^4-1}\,dx = \int -\frac{1}{2}\frac{1}{x^2+1} - \frac{1}{4}\frac{1}{x+1} + \frac{1}{4}\frac{1}{x-1}\,dx$$

$$= \underline{-\frac{1}{2}\arctan(x) - \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C}$$

oppgave 6

$$\int \frac{2x+1}{x^2-x-2}\,dx \qquad x^2+bx+c = (x+r)(x+s) = x^2+(r+s)x+s\cdot r$$

$$= \int \frac{2x+1}{(x-2)(x+1)}\,dx \qquad (-2)\cdot 1 = -2 \qquad -2+1 = -1$$

$$\frac{2x+1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$= \frac{A(x+1)}{(x-2)(x+1)} + \frac{B(x-2)}{(x-2)(x+1)}$$ Felles nevnere

Like tellere: $2x+1 = A(x+1) + B(x-2) = (A+B)x + A-2B.$

$\left(\begin{array}{l} 2 = A+B \\ 1 = A-2B \text{ etc} \ldots \end{array} \right)$

setter inn $x=-1$ $\quad 2(-1)+1 = 0 + B(-1-2)$

så $\underline{B = 1/3}$

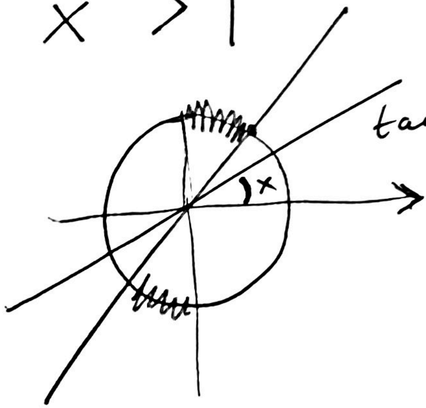
setter inn $x=2$ $\quad 5 = A\cdot 3 + 0$ så $\underline{A = 5/3}$

$$\int \frac{2x+1}{(x-2)(x+1)}\,dx = \int \frac{5}{3}\cdot\frac{1}{x-2} + \frac{1}{3}\cdot\frac{1}{x+1}\,dx = \underline{\frac{5}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + C}$$

spørsmål:

$\tan x > 1$

I

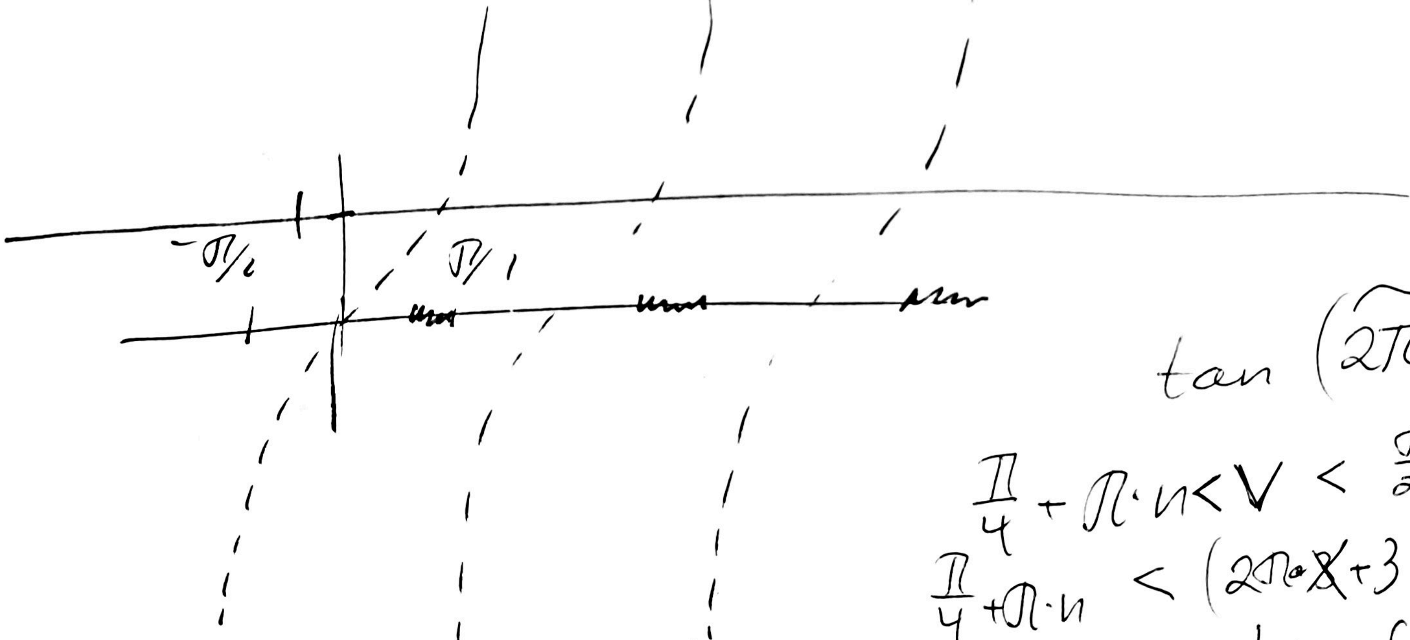


tan x: stigningstallet

$\frac{\pi}{4} + \pi \cdot n < x < \frac{\pi}{2} + \pi \cdot n$ $\qquad n \in \mathbb{Z}.$

$\bigcup_n \left\langle \frac{\pi}{4} + \pi \cdot n, \frac{\pi}{2} + \pi \cdot n \right\rangle$

II



$\tan(\overbrace{2\pi \cdot x + 3}^{V}) > 1$

$\frac{\pi}{4} + \pi \cdot n < V < \frac{\pi}{2} + \pi \cdot n$

$\frac{\pi}{4} + \pi \cdot n < (2\pi \cdot x + 3) < \frac{\pi}{2} + \pi \cdot n$

Løse for x:

$\frac{1}{2\pi}\left(\frac{\pi}{4} + \pi \cdot n - 3\right) < x < \frac{1}{2\pi}\left(\frac{\pi}{2} + \pi \cdot n - 3\right)$