23 mars
2021

Differensiallikninger (Diff likninger)

Likning som relaterer $y(x)$, $y'(x)$, $y''(x)$, ...
og variabelen x

	orden	
Eks.	1	$y' = f(x)$
	1	$y' = k \cdot y$ k konstant
	2	$y'' + ky = 0$
	2	$y'' + ay' + by = 0$
	1	$(y')^2 = q \cdot y$
	1	$y \cdot y' = e^x$

Orden til en diff. likning er høyste deriverte som forekommer
($y^{(n)}(x)$ n-te derivert av x)

5    $y^{(5)}(x) = x + 2$

Løsninger til diff. likninger er funksjoner $y(x)$ som oppfyller diff. likningen.

$$y'' + y = 0$$

$$(\sin x)' = \cos x$$

$$(\sin x)'' = (\cos x)' = -\sin x$$

$$(\cos x)'' = (-\sin x)' = -\cos x$$

Viser $y(x) = a \sin x + b \cos x$ for alle x er en løsning til $y'' + y = 0$.

To frie parametre.

En diff. likning har typisk mange løsninger

Antall "frie parametre" er typisk gitt ved ordenen til diff. likningen.

Betingelser på $y(x)$ slik at vi får en entydig løsning kalles randbetingelser.

$y'' + y = 0$

$\left.\begin{aligned} y(0) &= -3 \\ y'(0) &= 2 \end{aligned}\right\}$ randbetingelse

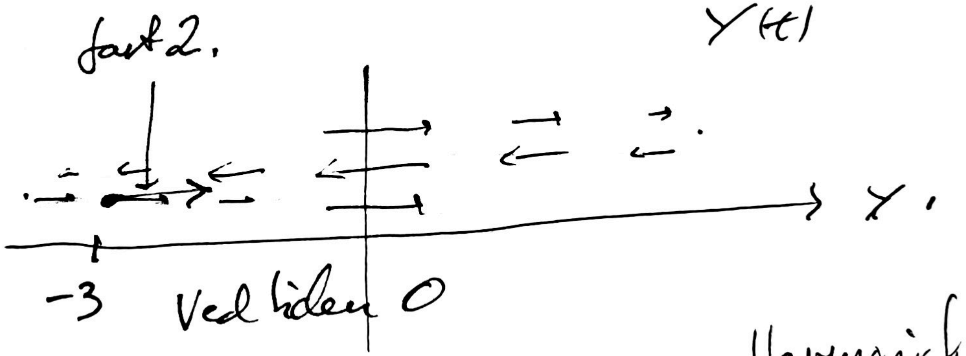
$y(x) = a\sin x + b\cos x$

$y'(x) = a\cos x - b\sin x$

$y(0) = \underline{-3 = b}$

$y'(0) = \underline{2 = a}$

Løsningen er $\underline{y(x) = 2\sin x - 3\cos x}$.



Ved tiden 0

$y(0) = -3$

$y'(t) = 2$.

Harmonisk svingning

$Y'(x) = f(x)$

Løsninger er antiderivert til $f(x)$ — én frihetsgrad.

$$\int f(x)\,dx = F(x) + C$$

$F(x)$ ← én antiderivert.

$$Y''(x) = x$$

$$(Y')' = x$$

$$Y' = \frac{x^2}{2} + C_1$$

$$Y(x) = \frac{x^3}{6} + C_1 \cdot x + C_2$$

2 frie variabler.

$$Y(0) = -3 = C_2$$

$$Y'(0) = 2 = C_1$$

Så $\underline{Y(x) = \frac{x^3}{6} + 2x - 3}$

(Matte 1000)

Eulers metode

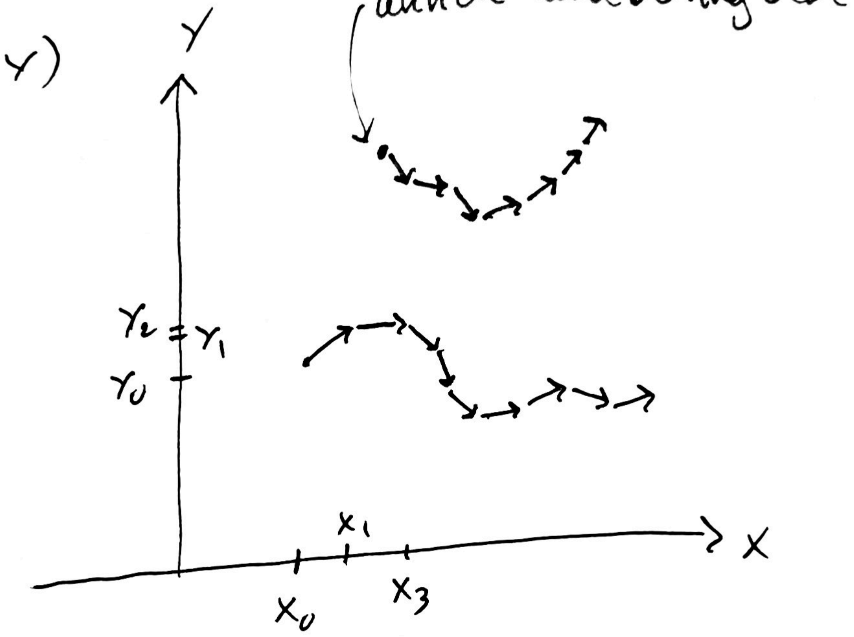
$$y' = F(x, y)$$

starter i

(x_0, y_0)

Randbetingelsen

$y(x_0) = y_0$

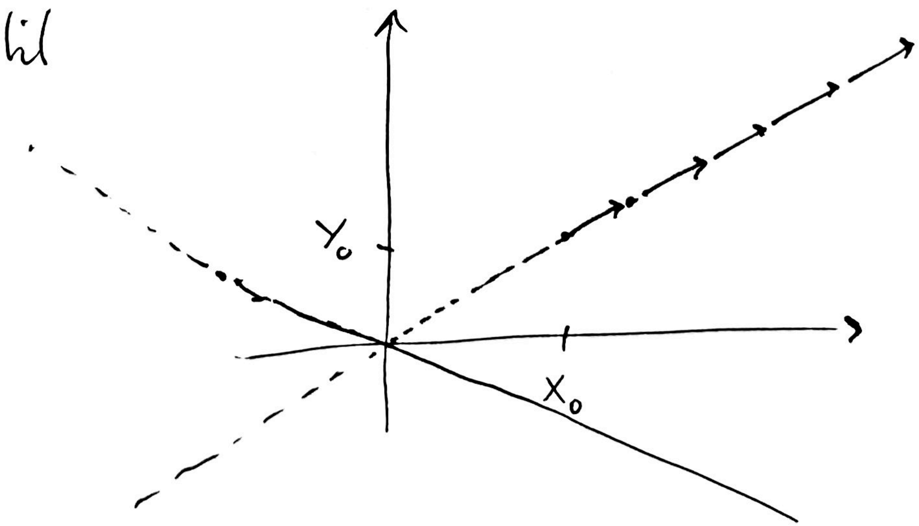


oppgave

Finn løsningen til

$$y' = \frac{y}{x}$$

løsningene er linjer gjennom origo.



$y' = \frac{y}{x}$ deler med y

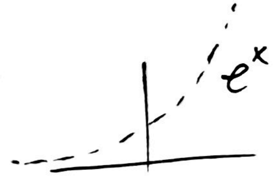
$\underbrace{\frac{y'}{y}}_{\text{bare } y} = \underbrace{\frac{1}{x}}_{\text{bare } x}$ eksempel på en separabel diff. likning.

$\int \frac{y'}{y}\,dx = \int \frac{1}{x}\,dx$ $\left(\begin{array}{c} y'dx \\ \frac{dy}{dx}dx \sim dy \\ \text{substitusjon} \end{array} \right)$

$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$

$\ln|y| = \ln|x| + C$

$|y| = e^{\ln|y|} = e^{\ln|x|+c} = e^c|x|$



$\left(e^c \text{ gir alle positive reelle tall} \right)$

$c \in \mathbb{R}$

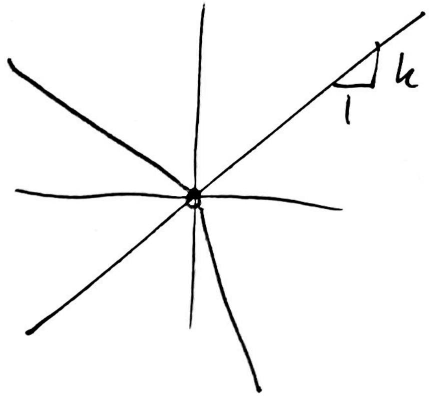
$y(x) = \pm e^c|x|$

$y(x) = k \cdot x$ $k \neq 0$.

Er $k=0$ også en løsning? $y \equiv 0$ og likningen er oppfylt.

$y(x) = kx \qquad k \in \mathbb{R}$

$y' = \frac{y}{x}$ ikke def for $x = 0$.



Hva er løsningene til $\quad y' = \frac{2y}{x}$?

prøver med $y = c \cdot x^2 \qquad y' = 2cx = 2 \cdot \frac{y}{x}$ ✓

$\frac{y}{x} = \frac{cx^2}{x} = c \cdot x$

$\int \frac{y'}{y} dx = \int 2 \cdot \frac{1}{x} dx$

$\ln|y| = 2 \cdot \ln|x| + k \qquad \left(\begin{aligned} &2\ln|x| \\ &= \ln|x|^2 \end{aligned} \right)$

$y = \pm e^k x^2$

$y = C x^2 \qquad C \in \mathbb{R}.$

$Y_{(t)}' = k\, Y_{(t)}$

Forenting (bank innskudd)

naturlig vekst

$y' = r \cdot y$

10% rente : $r = \frac{10}{100} = 0.1$

separabel diff likning

$$\int \frac{y'}{y}\, dt = \int r\, dt$$

$$e^{\ln|y|} = e^{(rt + c)}$$

$$|y| = e^{c} \cdot e^{rt}$$

$$\underline{y = k\, e^{rt}} \qquad k \in \mathbb{R}$$

$\left(k = \begin{matrix} 0 \checkmark \\ \pm e^{c} \ldots \end{matrix} \right)$

Separable diff. likninger er på formen

$$\underbrace{g(y)\, y'}_{\text{bare } y} = \underbrace{f(x)}_{\text{bare } x}$$

"vi kan separere x og y"

* $y' = e^{y} \cos x$ separabel $\frac{y'}{e^{y}} = e^{-y} y' = \cos x$

$y' = x^2 + y^2$ ($y' - y^2 = x^2$)

ikkje separabel

Hvordan løse separable diff likninger?

$$g(y)\, y' = f(x)$$

$$\int g(y) \underbrace{y' dx}_{dy} = \int f(x) dx$$

$$\underline{\int g(y) dy} = \int f(x) dx$$

oppg

$$y' = e^{y} \cos x$$

$$\int e^{-y} y' \, dx = \int \cos x \, dx$$

$$\int e^{-y} \, dy = \int \cos x \, dx$$

$u = -y$ $\qquad \int e^{u} (-du)$

$u' = -1$ $\qquad -\int e^{u} \, du$

$$-e^{-y} = \sin(x) + C_0 \qquad (C = -C_0)$$

$$e^{-y} = -\sin(x) + C$$

$$-y = \ln(C - \sin(x))$$

$$y(x) = \underline{-\ln(C - \sin(x))}$$

$$(1+x^2)\,y' = \frac{x}{y^5+1} \qquad \text{separabel}$$

$$(y^5+1)\,y' = \frac{x}{1+x^2}$$

$$u = 1+x^2$$
$$u' = 2x$$

$$\int (1+y^5)\underbrace{y'\,dx}_{dy} = \int \frac{x}{1+x^2}\,dx$$

$$= \int \frac{\frac{1}{2}u'}{u}\,dx$$

$$y + \frac{y^6}{6} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|1+x^2| + C$$

$$\frac{y^6}{6} + y = \frac{1}{2}\ln|1+x^2| + C$$

$$\underline{y^6 + 6\cdot y = 3\ln|1+x^2| + C}$$

$y(x)$ er gitt implisitt.
har ikkje en formel for $y(x)$.

oppgave

$$(y')^2 = 9y$$

$$y' = \pm 3\sqrt{y}$$

$$\frac{y'}{\sqrt{y}} = y^{-1/2} y' = \pm 3$$

$$\int y^{-1/2} \underbrace{y' dx}_{dy} = \int \pm 3 \, dx$$

$$\int y^{-1/2} dy = \pm 3x + C.$$

$$\frac{y^{1/2}}{1/2} = \pm 3x + c$$

$$\sqrt{y} = \pm \frac{3}{2} x + c$$

$$y = \left(c \pm \frac{3}{2}x\right)^2$$

$$= \left(c + \frac{3}{2}x\right)^2$$

Øving

Substitusjon

$$\int f(u(x))\, \overbrace{u'(x)\, dx}^{du} = \int f(u)\, du.$$

La $F'(u) = f(u)$ så $F(u)$ er en antiderivert til f.

$$\frac{d}{dx} F(u(x)) = f(u(x)) \cdot u'(x)$$

kjerneregelen

$$\int f(u(x))\, u'(x)\, dx = F(u(x)) + C$$

$$= \int f(u)\, du$$

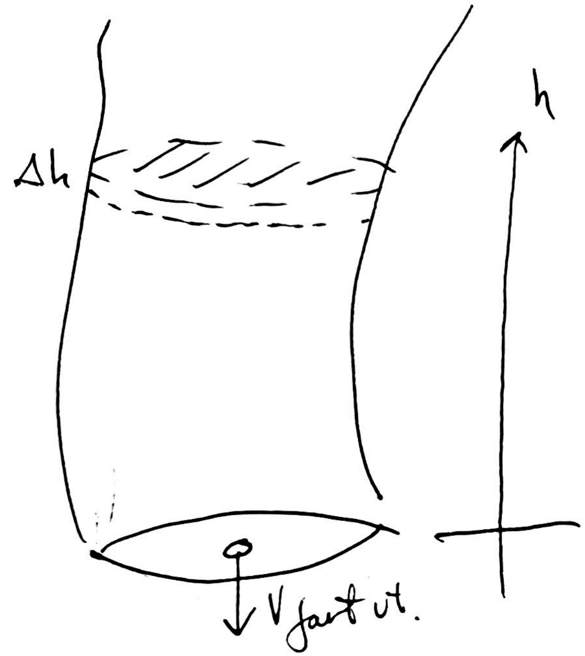
$u = 2 + x^3$

$u' = 3x^2$

$$\int \underbrace{3x^2}_{u'} \cdot \underbrace{(2+x^3)^7}_{u^7}\, dx$$

$$= \int u^7 u'\, dx = \int u^7\, du = \frac{u^8}{8} + C$$

$$= \frac{(2+x^3)^8}{8} + C$$



tverrsnitt areal til beholder ved høyde h : $A(h)$ (masse-tetthet ρ)

åpning areal a

$$\frac{dV}{dt} = A(h) \cdot h'(t)$$

$(\Delta V = A(h) \cdot \Delta h)$

$V(h)$ volum i beholder.

$$\Delta E = \underbrace{(\Delta V \cdot \rho)}_{\Delta m} \cdot g \cdot h = \frac{\overbrace{(\Delta V \cdot \rho)}^{\Delta m}}{2} v^2$$

v farten.

potensiell energi fall når ΔV av vasken renner ut

$v \cdot a \cdot \Delta t = -\Delta V$

$$-\frac{\Delta V}{\Delta t} \sim a \cdot v$$

$2gh = v^2$

$v = \sqrt{2gh}$

$$+A(h)\cdot h'(t) = -V'(t) = aV = a\sqrt{2gh}$$

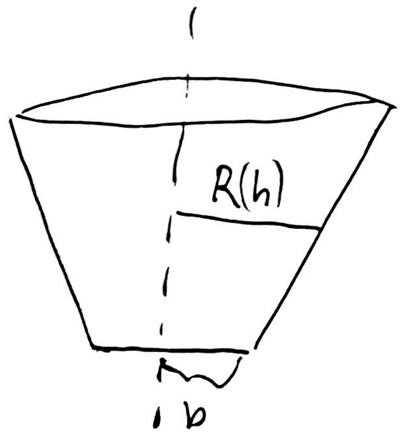
$$h'(t)\,A(h(t)) = -a\sqrt{2g\,h(t)}$$

$a \ll A(h)$

1. ordens diff. likning i $h(t)$

separabel $\qquad \dfrac{A(h)}{\sqrt{h}}\,h' = \underbrace{-a\sqrt{2g}}_{\text{konstant}}.$

Torricellis lov



$$R(h) = a + b\cdot h$$

$a = b = 1$ $\qquad R(h) = 1 + h.$

$$A(h) = \pi R(h)^2 = \pi(1+h)^2$$

$$\pi\,\frac{(1+h)^2}{\sqrt{h}}\,h' = -a\sqrt{2g}$$

$$\left(\frac{1+2h+h^2}{\sqrt{h}}\right)h' = -\frac{a\sqrt{2g}}{\pi}$$

$$\left(h^{-1/2}+2\sqrt{h}+h^{3/2}\right)h' = -\frac{a\sqrt{2g}}{\pi}$$

integrerer

$$\int h^{-1/2}+2\sqrt{h}+h^{3/2}\,dh = \int -\frac{a\sqrt{2g}}{\pi}\,dt$$

$$\frac{h^{1/2}}{1/2}+2\frac{h^{3/2}}{3/2}+\frac{h^{5/2}}{5/2} = \frac{-a\sqrt{2g}}{\pi}\cdot t + C$$

$$2\sqrt{h}+\frac{4}{3}h\sqrt{h}+\frac{1}{5}h^2\sqrt{h} = \frac{-a\sqrt{2g}}{\pi}t + c$$

$$\sqrt{h}\left(2+\frac{4}{3}h+\frac{h^2}{5}\right) = \ldots$$

implisitt beskrivelse av $h(t)$.

orden 3.

$$y''' = 2$$

Finn alle mulige løsninger y.

$$y''' = (y'')' = 2$$

$$y'' = 2x + C_1$$

$$(y')' = y'' = 2x + C_1$$

$$y' = x^2 + C_1 \cdot x + C_2$$

$$y(x) = \frac{x^3}{3} + \frac{C_1}{2} x^2 + C_2 x + C_3$$

3 parametre

$$y' = \frac{3x^2-4x}{2y+1} \quad \text{separabel}$$

$$\int (2y+1)\underbrace{y'dx}_{dy} = \int 3x^2-4x \, dx$$

$$y^2+y = x^3-2x^2+C.$$

Finn $y(x)$ og løs initialverdiproblemet $y(0) = 2$.

$y(0) = 2$

$$2^2+2 = 0 + C \quad \text{så} \quad C = 6$$

$$y^2+y = d$$

$$y^2+y-d = 0 \quad \text{2.grads likning.}$$

$$y = \frac{-1 \pm \sqrt{1-4(-d)}}{2} = \frac{-1 \pm \sqrt{1+4d}}{2}$$

$$y(x) = \frac{-1}{2} \pm \frac{1}{2}\sqrt{1+4(x^3-2x^2+c)}.$$

$\rightarrow$

$C = 6 \quad (y(0) = 2)$

$$y(x) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4(x^3 - 2x^2) + 24}$$

bare løsning med $y(0) = 2$ ved $+$.

$$\underline{y(x) = -\frac{1}{2} + \frac{1}{2}\sqrt{25 + 4(x^3 - 2x^2)}}$$

$$y' = ry\left(1 - \frac{y}{p}\right)$$ logistisk diff. likning.

oppgave $r = p = 1$:

$$y' = y(1-y)$$

$$\int \frac{y'}{y(1-y)}\,dx = \int 1\,dx$$

$$\int \frac{1}{y(1-y)}\,dy = x + C$$

Delbrøksoppspalting $\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$

Felles nevner gir: $1 = A(1-y) + By$

Så $A = B = 1$.

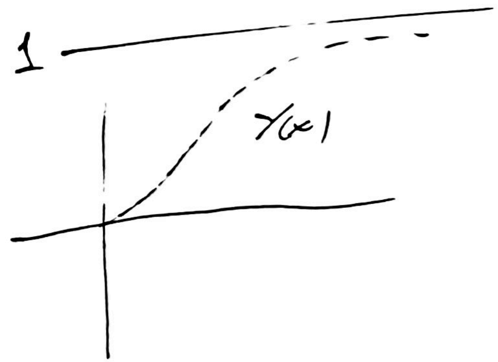
$u = 1-y$

$u' = -1$

$$\int \frac{1}{y(1-y)}\,dy = \int \frac{1}{y} + \frac{1}{1-y}\,dy$$

$$= \ln|y| - \ln|1-y| + \text{konst}$$

$$\underline{\ln\left|\frac{y}{1-y}\right| = x + c}$$



Løser for $y(x)$:

$$\left|\frac{y}{1-y}\right| = e^c e^x$$

$$\frac{y}{1-y} = k \cdot e^x$$

$$y = (ke^x)(1-y)$$

$$y(1 + ke^x) = ke^x$$

$$\underline{y(x) = \frac{ke^x}{1 + ke^x}} \quad \left(= \frac{k}{e^{-x} + k} \right)$$

$y' = y^2$ (vokser mye raskere enn $y' = y$ for $y > 1$)

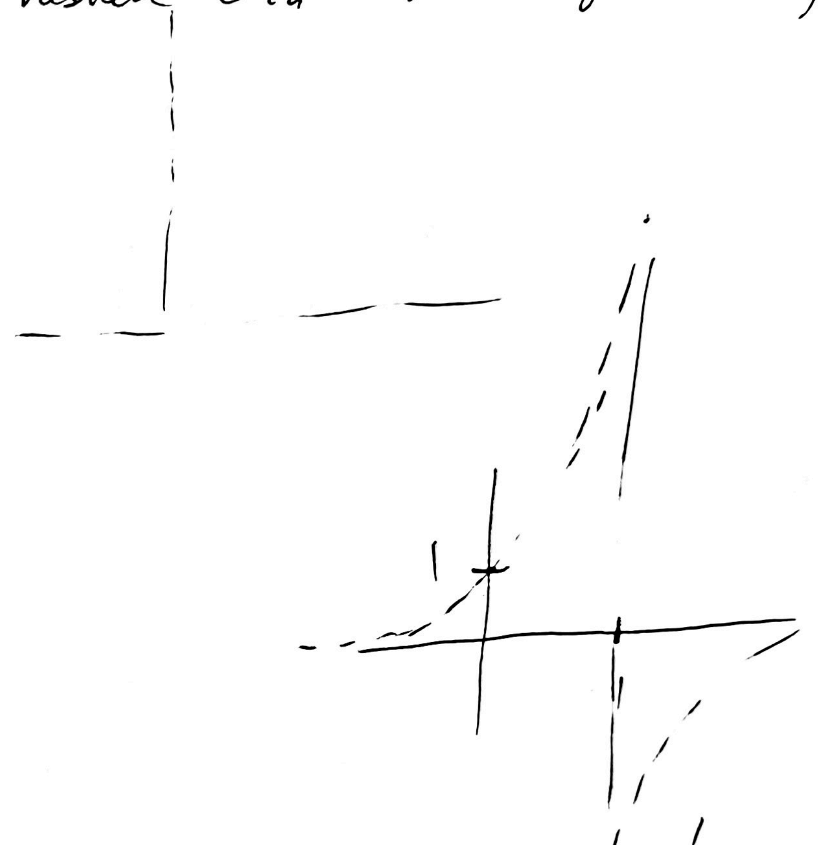
$$\frac{y'}{y^2} = 1$$

$$\int y^{-2} y' \, dx = \int 1 \, dx$$

$$\frac{y^{-1}}{-1} = x + C$$

$$\frac{1}{y} = C - x$$

$$\underline{y(x) = \frac{1}{C - x}}.$$



$$y(x) = \frac{1}{1-x}$$

Randbetingelse $y(0) = 1$, da må $C = 1$.

$y(x)$ går mot uendelig i endelig tid!

oppg

$$\cos(x) \cdot y'_{(x)} = e^{-2y} \sin x$$

$$e^{2y} y' = \frac{\sin x}{\cos x} = \tan x$$

$$\int e^{2y} y' dx = \int e^{2y} dy = \int \frac{\sin x}{\cos x} dx$$

$$\frac{1}{2} e^{2y} = -\ln|\cos x| + c$$

$$e^{2y} = (-2\ln|\cos x| + c)$$

anvender ln og deretter deler med 2

$$y(x) = \underline{\frac{1}{2} \ln (c - 2\ln|\cos x|)}$$

Forslag til oppg. fra dere:

$$\int \frac{x^2+9}{(x^2-1)(x^2+4)}\,dx = \int \frac{(x^2+4)+5}{(x^2-1)(x^2+4)}\,dx$$

$$\int \frac{1}{x^2-1}\,dx + 5\int \frac{1}{(x^2-1)(x^2+4)}\,dx$$

Delbrøksoppspalting

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$\frac{1}{(x^2-1)(x^2+4)} = \frac{1}{5}\left(\frac{1}{x^2-1} - \frac{1}{x^2+4}\right)$$

$$\text{int} = 2\int \frac{1}{x^2-1}\,dx - \int \frac{1}{x^2+4}\,dx$$

$u = \frac{x}{2}$

$u' = \frac{1}{2}$

$du = \frac{1}{2}dx$

$$= \int \frac{1}{x-1} - \frac{1}{x+1}\,dx - \int \frac{1}{4}\,\frac{1}{(1+(x/2)^2)}\,dx$$

$$\ln|x-1| - \ln|x+1| - \frac{1}{2}\int \frac{du}{1+u^2} = \ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$