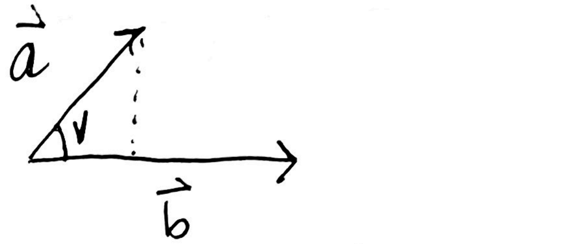
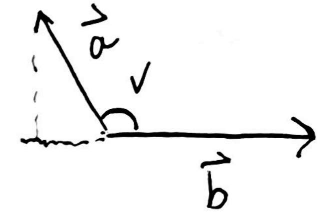
19.10
2021

13.4-6 Skalarproduktet

$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos(v)$





Egenskaper

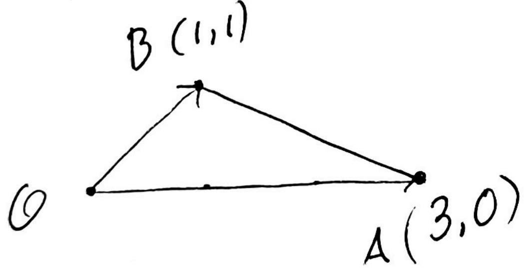
$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$(t\vec{a}) \cdot \vec{b} = t(\vec{a} \cdot \vec{b})$

Lineært i begge vektorene $(\vec{a}_1 + \vec{a}_2) \cdot \vec{b} = \vec{a}_1 \cdot \vec{b} + \vec{a}_2 \cdot \vec{b}$.

Skalarproduktet på koordinatform $[x_1, y_1] \cdot [x_2, y_2] = x_1 x_2 + y_1 \cdot y_2$

Eksempel



$\vec{OB} = [1,1]$ $\quad |\vec{OB}| = \sqrt{2}$

$\vec{OA} = [3,0]$ $\quad |\vec{OA}| = 3$

$\vec{AB} = \vec{OB} - \vec{OA}$
$= [1,1] - [3,0] = [-2,1]$

$|\vec{AB}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

$\angle O$:

$\vec{OA} \cdot \vec{OB} = |\vec{OA}| \cdot |\vec{OB}| \cos(\angle O)$

$[3,0] \cdot [1,1] = 3 \cdot 1 + 0 \cdot 1 = 3$

$\cos(\angle O) = \frac{3}{3 \cdot \sqrt{2}} = \underline{\frac{1}{\sqrt{2}}}$

$\underline{\angle O = 45^\circ}$

$$\cos(\angle A) = \frac{\vec{AB} \cdot \vec{AO}}{|\vec{AB}| \cdot |\vec{AO}|} = \frac{[-2,1] \cdot [-3,0]}{\sqrt{5} \cdot 3}$$

$$\left(\vec{AO} = -\vec{OA} = -[3,0] = [-3,0] \right)$$

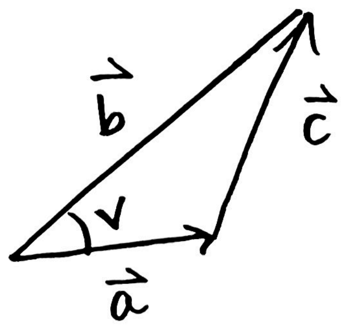
$$\cos(\angle A) = \frac{6+0}{3 \cdot \sqrt{5}} = \frac{2}{\sqrt{5}} \approx 0.8944$$

$$\angle A = 26.56^\circ$$

$$\angle B = 180^\circ - 45^\circ - 26.56^\circ = 135^\circ - 26.56^\circ$$

$$= \underline{108.44^\circ}$$

Utleder cossetningen ved å bruke skalarprodukt.



$|\vec{a}| = a$ $\quad |\vec{c}| = c$

$|\vec{b}| = b$

$$\vec{c} = \vec{b} - \vec{a}$$

$$\vec{c}\cdot\vec{c} = |\vec{c}|^2$$

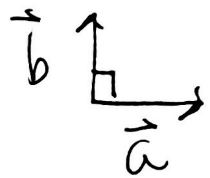
$$= (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a}) \qquad \cdot \text{ er lineært}$$

$$= \vec{b}\cdot(\vec{b}-\vec{a}) - \vec{a}\cdot(\vec{b}-\vec{a})$$

$$= \vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + (-\vec{a})\cdot(-\vec{a})$$

$$|\vec{c}|^2 = |\vec{b}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{a}|^2$$

$$\underline{c^2 = a^2 + b^2 - 2a\cdot b\cos(V)}$$

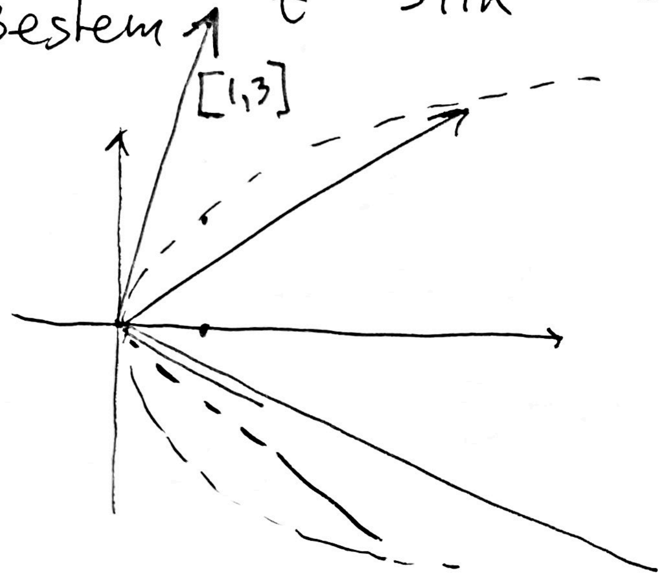


vinkelrette vektorer $\vec{a}, \vec{b} \neq \vec{0}$

$\vec{a}\cdot\vec{b} = 0$ ortogonale

$|\vec{a}||\vec{b}|\cos(V)$

Bestem t slik at $[t^2, t]$ og $[1,3]$ er ortogonale.



$$[t^2, t] \cdot [1,3] = 0$$

$$t^2 + 3t = 0$$

$$t(t+3) = 0$$

$t = 0$ eller $t+3 = 0$

$t = -3$.

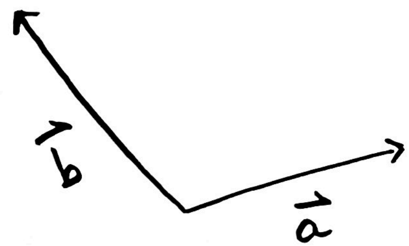
To løsninger

$t = 0$: $[0,0]$

$t = -3$: $\underline{[9,-3]}$

$|\vec{a}| = 5, \quad |\vec{b}| = 7 \qquad \vec{a}\cdot\vec{b} = -10$

$\vec{u} = 2\vec{a} + 3\vec{b} \qquad \vec{v} = (\vec{b} - 3\vec{a})$



Hva er $\vec{u}\cdot\vec{v}$?

$(2\vec{a} + 3\vec{b})\cdot(\vec{b} - 3\vec{a})$ benytter linearitet til skalarproduktet.

$(2\vec{a})\cdot\vec{b} + (2\vec{a})\cdot(-3\vec{a}) + (3\vec{b})\cdot\vec{b} + (3\vec{b})\cdot(-3\vec{a})$

$2(-3)\,\vec{a}\cdot\vec{a} + 3\,\vec{b}\cdot\vec{b} + 2\,\vec{a}\cdot\vec{b} + 3(-3)\,\vec{a}\cdot\vec{b}$

$\vec{u}\cdot\vec{v} = -6|\vec{a}|^2 + 3|\vec{b}|^2 - 7\,\vec{a}\cdot\vec{b}$

$= -6\cdot 5^2 + 3\cdot 7^2 - 7(-10)$

$-150 \quad (150-3) \quad + 70 \quad = \underline{67}$

$$|\vec{u}|^2 = \vec{u}\cdot\vec{u} = (2\vec{a}+3\vec{b})\cdot(2\vec{a}+3\vec{b})$$

$$= 4\vec{a}\cdot\vec{a} + 6\vec{a}\cdot\vec{b} + 6\vec{a}\cdot\vec{b} + 9\vec{b}\cdot\vec{b}$$

$$= 4|\vec{a}|^2 + 9|\vec{b}|^2 + 12\vec{a}\cdot\vec{b}$$

$$= 4\cdot 5^2 + 9\cdot 7^2 + 12(-10)$$

$$= 100 \quad 450-9 \quad -120 = \underline{421}$$

$$|\vec{u}| = \sqrt{421}.$$

$$|\vec{v}|^2 = \vec{v}\cdot\vec{v} = (\vec{b}-3\vec{a})(\vec{b}-3\vec{a})$$

$$= \vec{b}\cdot\vec{b} - 3\vec{a}\cdot\vec{b} - 3\vec{a}\vec{b} + 9\vec{a}\cdot\vec{a}$$

$$= 9|\vec{a}|^2 + |\vec{b}|^2 - 6\vec{a}\cdot\vec{b}$$

$$= 9\cdot 25 + 49 - 6(-10)$$

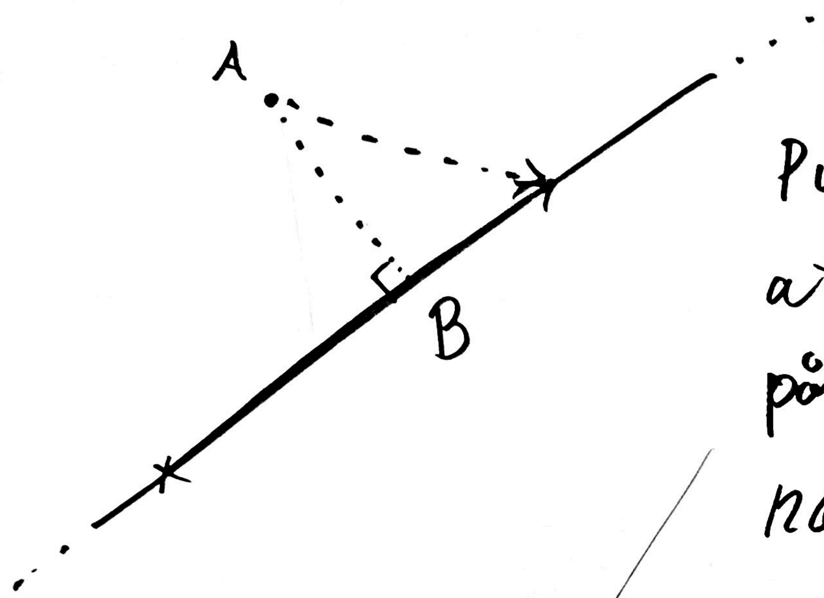
$$= 250 + 24 + 60 = 334$$

$$|\vec{v}| = \sqrt{334}$$

La w være vinkelen mellen $\vec{u}$ og $\vec{v}$

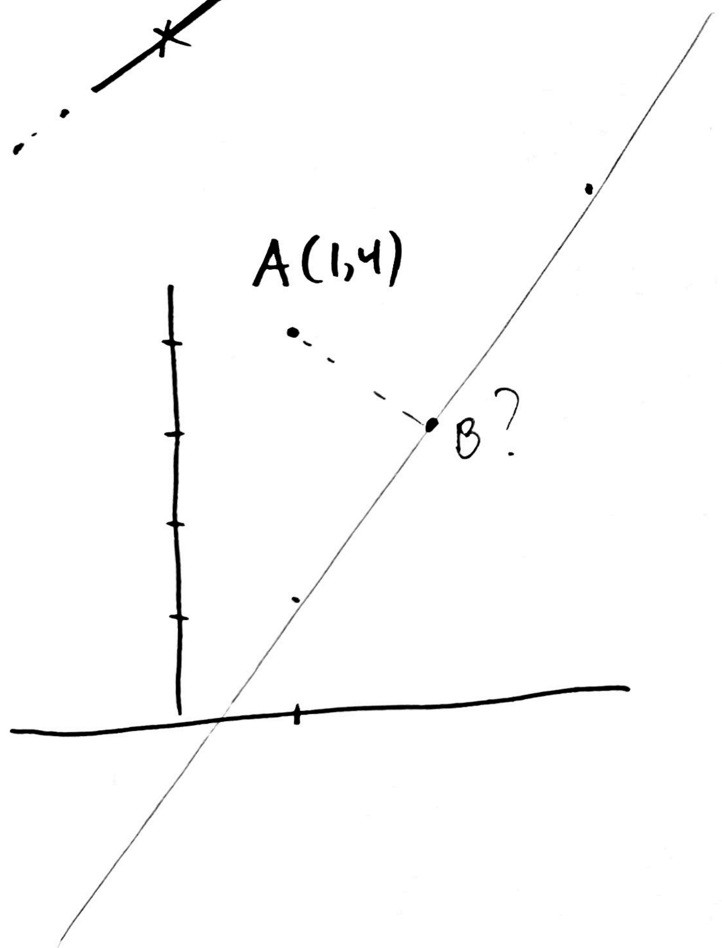
$$\cos(w) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{67}{\sqrt{421}\sqrt{334}} = f$$

$$W = \arccos(f) = \underline{79.707}.$$



Punktet B på linjen slik at $\vec{AB}$ står vinkelrett på linjen er punktet nærmest A (på linjen)

Eks.



Parametrisert linje

$$\begin{cases} x = 3t+1 \\ y = 4t+1 \end{cases}$$

Finn et punkt B på linjen slik $\vec{AB} \perp$ på linjen.

$$\overrightarrow{O(x,y)} = [3t+1,\ 4t+1] = t\underbrace{[3,4]}_{\text{retningsvektor}} + [1,1]$$

$$\overrightarrow{A(x,y)} = \overrightarrow{O(x,y)} - \overrightarrow{OA} = t[3,4] + [1,1] - [1,4]$$
$$= t[3,4] + [0,-3].$$

$$[3,4] \bullet \overrightarrow{A(x,y)} = 0$$
$$[3,4] \bullet \left(t[3,4] + [0,-3] \right) = 0$$
$$t\,|[3,4]|^2 + [3,4] \bullet [0,-3] = 0$$
$$25 \cdot t - 12 = 0$$
$$\underline{t = \frac{12}{25}}$$

koordinater til B :

$$x = 1 + 3 \cdot \frac{12}{25} = \frac{25+36}{25} = \frac{61}{25}$$
$$y = 1 + 4 \cdot \frac{12}{25} = \frac{25+48}{25} = \frac{73}{25}$$

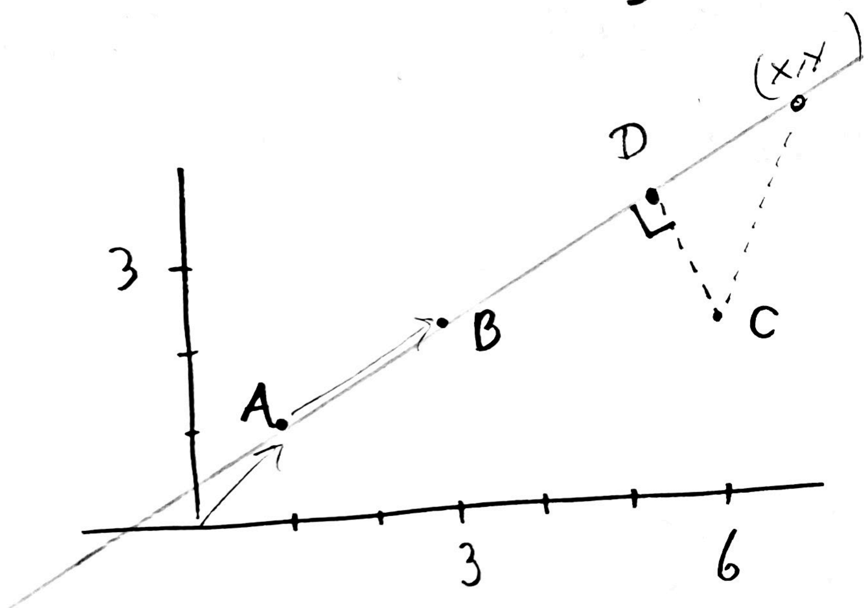
$$\underline{B\left(\frac{61}{25}, \frac{73}{25}\right)}$$

oppg

$A(1,1)$ $B(3,2)$

$C(6,2)$

Finn punktet på linjen gjennom A, B som er nærmest C.



Retningsvektor for linjen gjennom A, B er

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= [3,2] - [1,1]$

$= [2,1]$

parametrisering av linjen

$\overrightarrow{O(x,y)} = [x,y] = \overrightarrow{OA} + t\overrightarrow{AB}$

$[x,y] = [1,1] + t[2,1]$

$\overrightarrow{C(x,y)} = \overrightarrow{O(x,y)} - \overrightarrow{OC} = [1,1] + t[2,1] - [6,2]$

$$\overrightarrow{C(x,y)} = [-5,-1] + t[2,1]$$

$$\overrightarrow{AB} \bullet \overrightarrow{C(x,y)} = 0$$

$$[2,1] \cdot ([-5,-1] + t[2,1]) = 0$$

$$[2,1] \cdot [-5,-1] + |[2,1]|^2 \cdot t = 0$$

$$(-10-1) + 5t = 0$$

$$5t = 11$$

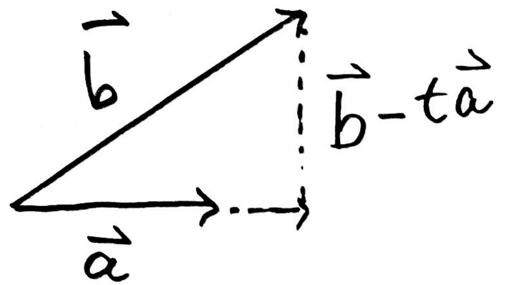
$$t = \frac{11}{5}$$

Dette gir

$$x = 1 + 2t = 1 + \frac{22}{5} = \frac{27}{5}$$

$$y = 1 + t = 1 + \frac{11}{5} = \frac{16}{5}$$

$$D\left(\frac{27}{5}, \frac{16}{5}\right) \quad (\sim (5,3))$$



komponenten til $\vec{b}$ langs $\vec{a}$.

$t\cdot\vec{a}$ slik at

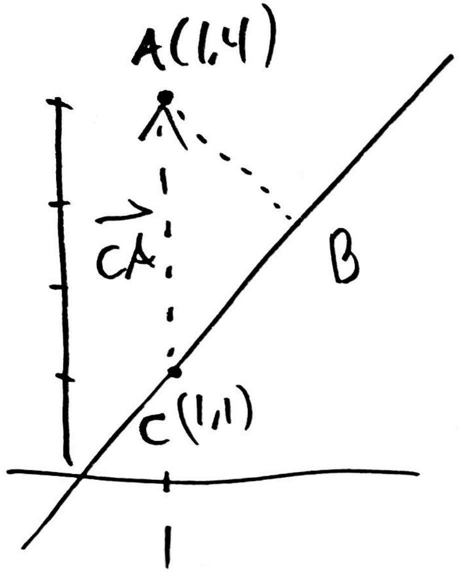
$$(\vec{b}-t\vec{a})\cdot\vec{a}=0$$

$$\vec{b}\cdot\vec{a}-t\,\vec{a}\cdot\vec{a}=0$$

$$t=\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}$$

$$\vec{b}_{\parallel}=\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$$

$$\vec{b}_{\perp}=\vec{b}-\vec{b}_{\parallel}=\underline{\vec{b}-\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}}$$



retningsvektor $[3,4] = \vec{r}$

Finn B nærmest A
på linjen

Løser problemet (fra forelesningen)
ved å regne ut komponenten
til $\vec{CA}$ langs $\vec{r}$.

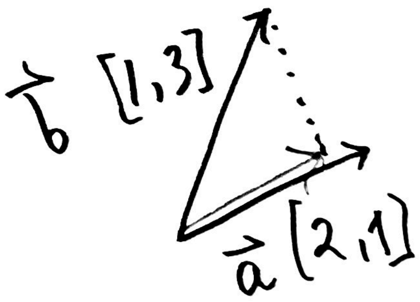
Vi finner
$B\left(\frac{61}{25}, \frac{73}{25}\right)$

$$\vec{CA} = \vec{OA} - \vec{OC} = [1,4] - [1,1] = [0,3].$$

$$\vec{CB} = \vec{CA}_{\parallel \vec{r}} = \frac{\vec{CA} \cdot \vec{r}}{|\vec{r}|^2}\vec{r}$$

$$= \frac{[0.3] \cdot [3.4]}{|[3.4]|^2}[3.4] = \frac{12}{25}[3,4]$$

$$\vec{OB} = \vec{OC} + \vec{CB} = [1,1] + \left[\frac{36}{25}, \frac{48}{25}\right] = \left[\frac{61}{25}, \frac{73}{25}\right]$$



$$\vec{b}_{\parallel \vec{a}} = \frac{\vec{b} \bullet \vec{a}}{|\vec{a}|^2} \vec{a}$$

$$= \frac{[1,3] \bullet [2,1]}{|[2,1]|^2} [2,1]$$

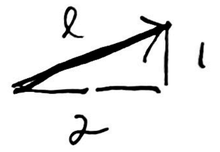
$$= \frac{5}{5} [2,1] = \underline{[2,1]}$$

$$[1,3] \bullet [2,1] = 1 \cdot 2 + 3 \cdot 1 = 5$$

$$|[2,1]|^2 = (\sqrt{5})^2 = 5$$

$$[x_1, y_1] \bullet [x_2, y_2] = x_1 \cdot x_2 + y_1 \cdot y_2$$

SKALAR (tall)



Pytagoras

$$1^2 + 2^2 = l^2$$

$$5 = l^2$$

$$l = \sqrt{5}$$

vektorer $\vec{a}$  størrelse $|\vec{a}|$

og retning.

$\frac{\vec{a}}{|\vec{a}|}$ "peker" i samme retning som $\vec{a}$

: har samme retning som $\vec{a}$

$\frac{\vec{a}}{|\vec{a}|}$ har lengde 1

$$\left| \frac{1}{|\vec{a}|} \vec{a} \right| = \frac{1}{|\vec{a}|} \cdot |\vec{a}| = 1.$$

enhetsvektor

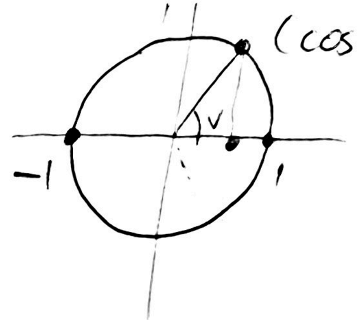
lengde lik 1.

$$\vec{a} = \underbrace{|\vec{a}|}_{\text{størrelsen}} \cdot \frac{\vec{a}}{|\vec{a}|}$$

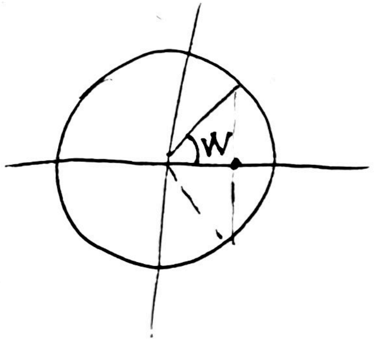
retningen til $\vec{a}$

$\vec{a} \neq 0$

Variant oblig 2 b) $\cos v = -1$ $v \in [0, 2\pi]$



Én løsning $\underline{v = \pi}$



$\cos v = 0.7$

$v \in [0°, 360°]$

(forventer $v \sim 45°$ litt større)

$$w = \overbrace{\arccos(0.7)}^{45.57°}.$$

Alle løsninger $v = w + 360°$

$= -w + 360°.$

i $[0°, 360°]$ er $w,\ 360° - w$

$\underline{45.57°}$ og $\underline{314.43°}$

Eksakt: $\underline{\arccos(0.7)}$ og $\underline{360° - \arccos(0.7)}$