

27.10  
2021

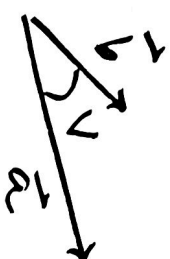
# 14.5 Vektorprodukt (kryssprodukt)

$\vec{a} \times \vec{b}$  vektor i  $\mathbb{R}^3$

① Definer for  $\vec{a}, \vec{b}$  vektorer i  $\mathbb{R}^3$ .

Størrelsen:

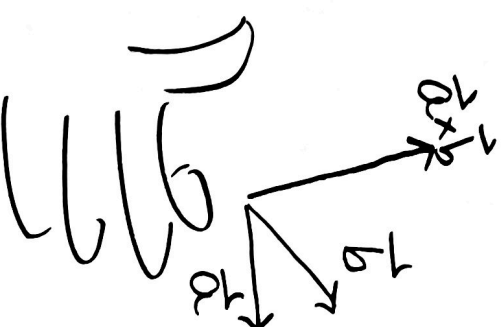
$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin \varphi$$



$$0 \leq \varphi \leq 180^\circ$$

retningen:

$\vec{a} \times \vec{b}$  vinkelret på  $\vec{a}$  og  $\vec{b}$   
 $\vec{a}, \vec{b}$  og  $\vec{a} \times \vec{b}$  er højrehåndssystem

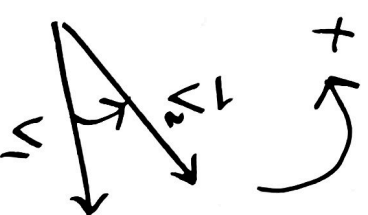


Kreuzprodukt für Ebene i xy-Planes.

$$[x_1, y_1, 0] \times [x_2, y_2, 0] \\ (x_1 \vec{e} + y_1 \vec{j}) \times (x_2 \vec{e} + y_2 \vec{j}) = (\text{linear combination}) \\ x_1 x_2 \underbrace{\vec{e} \times \vec{e}}_{\vec{0}} + x_1 y_2 \underbrace{\vec{e} \times \vec{j}}_{-\vec{i}} + x_2 y_1 \underbrace{\vec{j} \times \vec{e}}_{\vec{i}} + y_1 y_2 \underbrace{\vec{j} \times \vec{j}}_{\vec{0}}$$

$$= (x_1 y_2 - x_2 y_1) \underbrace{\vec{e} \times \vec{j}}_{\vec{k}} = [0, 0, x_1 y_2 - x_2 y_1] \\ = [0, 0, |x_1 \ y_1 \mid x_2 \ y_2 \mid|]$$

②



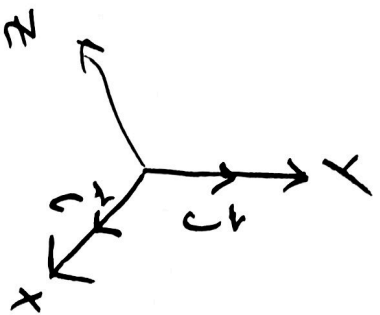
$$\det \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} > 0$$

$\Leftrightarrow$  Koefizienten  
von Basisvektoren  
 $v_1$  und  $v_2$  positiv

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b} \quad \text{antikommutativt.}$$

$$\vec{a} \times \vec{a} = \vec{0}$$

③



$$\vec{e}_1 = \vec{i}$$

$$\vec{e}_2 = \vec{j}$$

$$\vec{e}_3 = \vec{k}$$

$$\vec{i} \times \vec{j} = \vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i}$$

$$\vec{k} \times \vec{i} = -\vec{j}$$

Kryssproduktet er lineart i begge vektorene

$$s(\vec{a}) \times \vec{b} = s(\vec{a} \times \vec{b})$$

$$(\vec{a}_1 + \vec{a}_2) \times \vec{b} = \vec{a}_1 \times \vec{b} + \vec{a}_2 \times \vec{b}$$

Se notater under "file" i Canvas for detaljer.

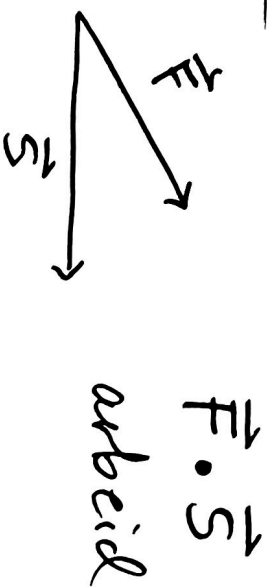
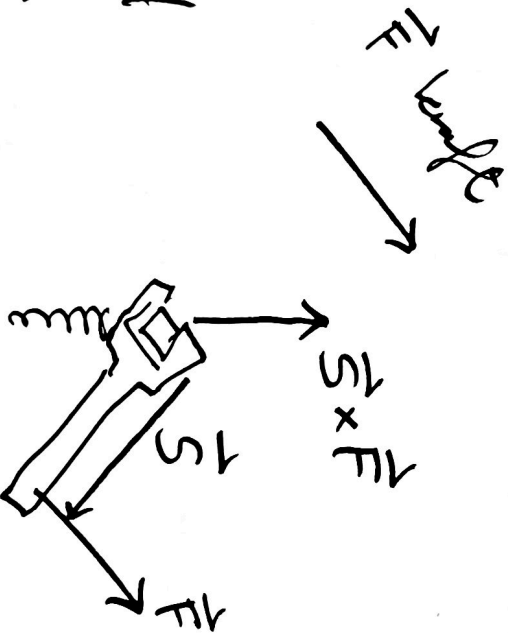
$$\begin{aligned}
 [1, 2, 0] \times [3, -4, 0] &= [0, 0, |1, 2 \\
 &= [0, 0, 1(-4) - 2 \cdot 3] = [0, 0, -10] = -10[0, 0, 1] \\
 &= -10\vec{k}
 \end{aligned}$$

④

$$\begin{aligned}
 (2\vec{a} + \vec{b}) \times (\vec{a} - 3\vec{b}) &\stackrel{\text{lineær}}{=} \underbrace{2\vec{a} \times \vec{a}}_0 - 6\vec{a} \times \vec{b} + \underbrace{\vec{b} \times \vec{a}}_{-\vec{a} \times \vec{b}} - 3\underbrace{\vec{b} \times \vec{b}}_0 \\
 &= -7\vec{a} \times \vec{b}
 \end{aligned}$$

Fysikk:

Dreiemoment  
 $\vec{S} \times \vec{F}$



Kryssprodukt  
 vinkel i Maxwell's  
 ligninger.

Vektorprodukt in Koordinaten:

$$[x_1, y_1, z_1] \times [x_2, y_2, z_2] = (x_1 \vec{e} + y_1 \vec{j} + z_1 \vec{k}) \times (x_2 \vec{e} + y_2 \vec{j} + z_2 \vec{k}) =$$

$$\begin{aligned} & \textcircled{5} \quad x_1 x_2 \underbrace{\vec{e} \times \vec{e}}_{\vec{0}} + x_1 y_2 \underbrace{\vec{e} \times \vec{j}}_{\vec{k}} + x_1 z_2 \underbrace{\vec{e} \times \vec{k}}_{-\vec{j}} + y_1 x_2 \underbrace{\vec{j} \times \vec{e}}_{-\vec{k}} + y_1 y_2 \underbrace{\vec{j} \times \vec{j}}_{\vec{0}} + y_1 z_2 \underbrace{\vec{j} \times \vec{k}}_{\vec{e}} + z_1 x_2 \underbrace{\vec{k} \times \vec{e}}_{\vec{j}} + z_1 y_2 \underbrace{\vec{k} \times \vec{j}}_{-\vec{e}} + z_1 z_2 \underbrace{\vec{k} \times \vec{k}}_{\vec{0}} \\ & = (x_1 y_2 - y_1 x_2) \underbrace{\vec{e} \times \vec{j}}_{\vec{k}} + (x_1 z_2 - z_1 x_2) \underbrace{\vec{e} \times \vec{k}}_{-\vec{j}} \\ & \quad + (y_1 z_2 - z_1 y_2) \underbrace{\vec{j} \times \vec{k}}_{\vec{e}} \end{aligned}$$

$$= [y_1 z_2 - z_1 y_2, -(x_1 z_2 - z_1 x_2), (x_1 y_2 - y_1 x_2)]$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}, - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \Bigg]$$

3-Determinanten

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

⑥

$$= a_1 \overline{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}} - a_2 \overline{\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}} + a_3 \overline{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}$$

$$+ \quad - \quad +$$

$$\begin{vmatrix} 1 & 2 & -2 \\ 1 & 4 & 5 \\ -3 & 0 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 5 \\ 0 & 2 \end{vmatrix} \overset{\text{minus}}{\leftarrow} -2 \cdot \begin{vmatrix} 1 & 5 \\ -3 & 2 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 1 & 4 \\ -3 & 0 \end{vmatrix}$$

$$= 8 - 2(17) + (-2)(0 - (-12))$$

$$= 8 - 34 - 24 = -50$$

Kryssprodukt udflydt ved 3-determinant

$$[x_1, y_1, z_1] \times [x_2, y_2, z_2] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

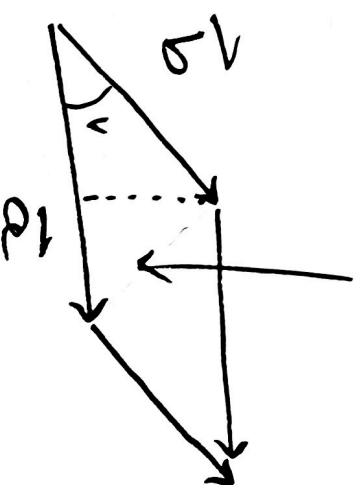
⑦

$$[1, 0, 2] \times [1, 1, -1] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \underline{\underline{[-2, 3, 1]}}$$

areal  $|\vec{a} \times \vec{b}|$



Oppg.

1)

Finne kryssproduktet

$$\vec{OA} \times \vec{OB}$$

$$A(3, -1, 2)$$

$$B(1, 3, 4)$$

8

2) Finne arealet til trekanten med hjørner O, A og B.

$$\vec{OA} = [3, -1, 2]$$

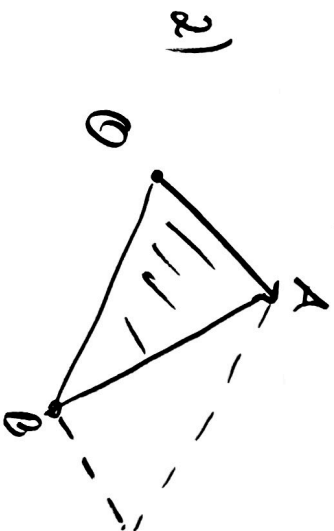
og

$$\vec{OB} = [1, 3, 4]$$

$$\vec{OA} \times \vec{OB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 3 & 1 & 3 \end{vmatrix}$$

$$= [-4 - 6, -(12 - 2), 9 - (-1)] = 10[-1, -1, 1]$$



Arealet trekant OAB

$$= \frac{1}{2} |\vec{OA} \times \vec{OB}| = \frac{1}{2} \cdot 10 \sqrt{(-1)^2 + (-1)^2 + 1^2}$$

$$= \underline{5\sqrt{3}}$$

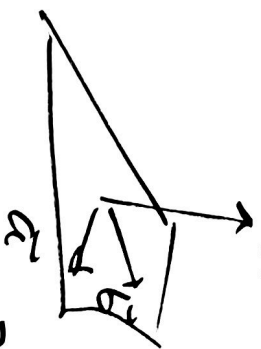
Finne alle vektorer som står vinkelrett

på planet som inneholder vektorene

$$\vec{a} = [1, 1, -3] \text{ og } \vec{b} = [2, 3, 4].$$

(9)

En slik vektor er  $\vec{a} \times \vec{b}$ .



$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ 2 & 3 & 4 \end{vmatrix} = \begin{bmatrix} |1 \cdot 4 - (-3) \cdot 3| & -|1 \cdot 4 - (-3) \cdot 2| & |1 \cdot 3 - 1 \cdot 2| \end{bmatrix}$$

$$= [13, -10, 1]$$

(sjekker at  $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$   
 $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ )

Alle vektorer vinkelrett på planet er parallelle.

Vektorene vinkelrett på planet er derfor

$$t[13, -10, 1] \text{ for } t \neq 0.$$