1 nov
2021
Faush

14.6 og 7 Trevektorprodukt
Likning for et plan.

$$\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = 1\cdot 4 - (2)(-3) = 4-(-6) = 10$$

$$\begin{array}{ccc} + & - & + \\ \end{array}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (45-48) - 2(36-42) + 3(32-35)$$

$$= -3 - 2(-6) + 3(-3)$$

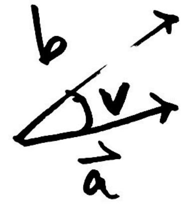
$$= -3 + 12 - 9 = \underline{0}$$

1

Kryssprodukt

$\vec{a} \times \vec{b}$ 3-vektor

$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin(v)$



$\vec{a} \times \vec{b}$ vinkelrett på $\vec{a}, \vec{b}$

$\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ høyrehansystem.

2

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i}, & \vec{j}, & \vec{k} \\ & \vec{a} & \\ & \vec{b} & \end{vmatrix}$$

$\vec{i} \times \vec{j} = \vec{k}$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = k$$

$\vec{i} \times \vec{k} = -\vec{j}$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\vec{j} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -\vec{j}$$

$\vec{j} \times \vec{k} = \vec{i}$

Determinanter er lineær i hver rad (og søyle)
skifter fortegn ved bytte av to rader.

(3)

$$\begin{vmatrix} 5 & 25 \\ 3 & -6 \end{vmatrix} = \begin{vmatrix} 5[1,5] \\ 3[1,-2] \end{vmatrix} = 3\cdot 5 \begin{vmatrix} 1 & 5 \\ 1 & -2 \end{vmatrix}$$

$$= 15(-2-5) = -15\cdot 7 = -(10+5)\cdot 7 = \underline{-105}$$

oppg Regn ut

$$\begin{array}{ccc} + & - & + \\ \end{array}$$

$$\begin{vmatrix} 10 & 20 & 40 \\ 3 & 9 & 12 \\ 14 & -7 & 21 \end{vmatrix} = \begin{vmatrix} 10[1,2,4] \\ 3[1,3,4] \\ 7[2,-1,3] \end{vmatrix}$$

$$= 3\cdot 7\cdot 10 \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \\ 2 & -1 & 3 \end{vmatrix} = 3\cdot 7\cdot 10 \left(1 \begin{vmatrix} 3 & 4 \\ -1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} + 4 \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \right)$$

$$(1(13) - 2(-5) + 4(-1-6))$$

$$= 3\cdot 7\cdot 10 (\underbrace{13+10-28}_{-5}) = -3\cdot 5\cdot 7\cdot 10 = -3\cdot \underbrace{35\cdot 10}_{-105} = \underline{-1050}$$

(4)

$$\begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix} = \begin{vmatrix} \vec{a} \\ \vec{b} + k\vec{a} \\ \vec{c} \end{vmatrix} = \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix} + k \underbrace{\begin{vmatrix} \vec{a} \\ \vec{a} \\ \vec{c} \end{vmatrix}}_{0}$$

to like rader!

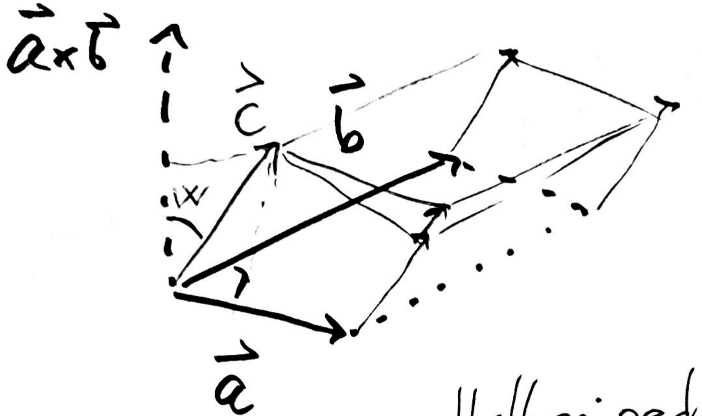
fordi: ved linearitet

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} [1,2,4] \\ [1,3,4]-[1,2,4] \\ [2,-1,3] \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix}$$

bytter

$$= -\begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & 4 \\ 2 & -1 & 3 \end{vmatrix} = -\left(\underbrace{0 \cdot \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix}}_{0} - 1\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} + \underbrace{0\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}}_{0}\right)$$

$$= +\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 1 \cdot 3 - 2 \cdot 4 = 3 - 8 = \underline{-5}$$



høyden
$|\vec{c}|\cdot\cos(w)$

$|\vec{a}\times\vec{b}|$ = arealet til parallellogrammet gitt ved $\vec{a}$ og $\vec{b}$

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parallellepipeder utspent av $\vec{a}, \vec{b}$ og $\vec{c}$ har volum

$$V = \underbrace{|\vec{c}|\,|\cos w|}_{\text{høyden}} \cdot \underbrace{|\vec{a}\times\vec{b}|}_{\text{grunnflate}}$$

$$V = |\vec{c}\cdot(\vec{a}\times\vec{b})|$$

$$= [c_1, c_2, c_3] \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ & \vec{a} & \\ & \vec{b} & \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ & \vec{a} & \\ & \vec{b} & \end{vmatrix} = \begin{vmatrix} \vec{c} \\ \vec{a} \\ \vec{b} \end{vmatrix} = -\underbrace{\begin{vmatrix} \vec{a} \\ \vec{c} \\ \vec{b} \end{vmatrix}}_{\text{bytter } \vec{a} \text{ og } \vec{c}}$$

$$= \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix}$$

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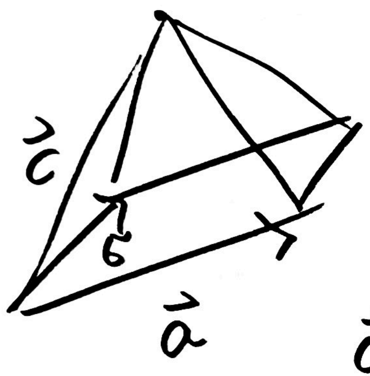
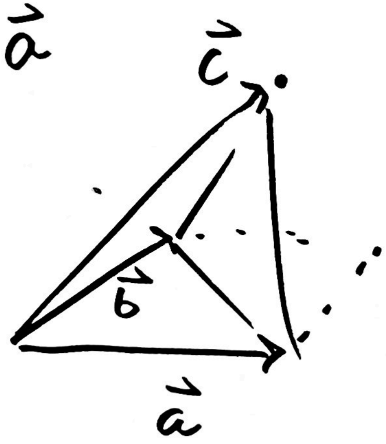
$$\begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix} = \vec{a} \cdot (\vec{b} \times \vec{c})$$ kalles også trippelproduktet.

Volum  $= \frac{1}{3}$ volum parallellepipeden.

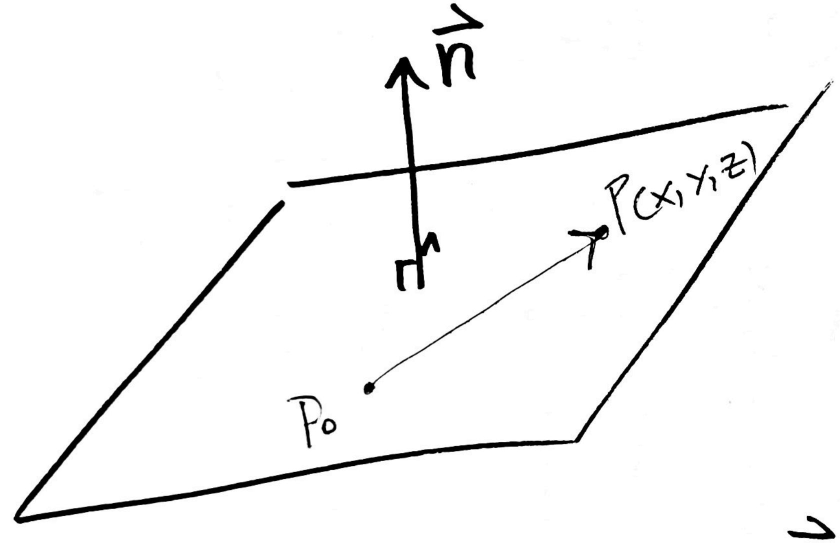
Volum  $= \frac{1}{6}$ volum parallellepiden.

7

14.7 Likning for et plan i $\mathbb{R}^3$

$$ax + by + cz = d$$

$\vec{n} = [a, b, c]$ er vinkelrett til planet



$$\overrightarrow{P_0P} \bullet \vec{n} = 0$$

$$\Updownarrow$$

$P(x, y, z)$ ligger i planet

$P_0(x_0, y_0, z_0)$, $\vec{n} = [a, b, c]$

$$\overrightarrow{P_0P} \bullet \vec{n} = (\overrightarrow{OP} - \overrightarrow{OP_0}) \bullet \vec{n}$$

$$= ([x, y, z] - [x_0, y_0, z_0]) \bullet [a, b, c] = 0$$

gir $[x, y, z] \bullet [a, b, c] = [x_0, y_0, z_0] \bullet [a, b, c]$

8

$$ax + by + cz = d \qquad \left(\begin{array}{l} d = 0 \Leftrightarrow \\ O \text{ er i planet} \end{array} \right)$$

Eks Beskriv planet som inneholder punktene O, $A(1,2,-1)$ og $B(2,0,3)$ ved en likning.

$\vec{a} = \overrightarrow{OA} = [1,2,-1]$ $\qquad \vec{b} = \overrightarrow{OB} = [2,0,3]$.

$\vec{a} \times \vec{b}$ står vinkelrett på planet.

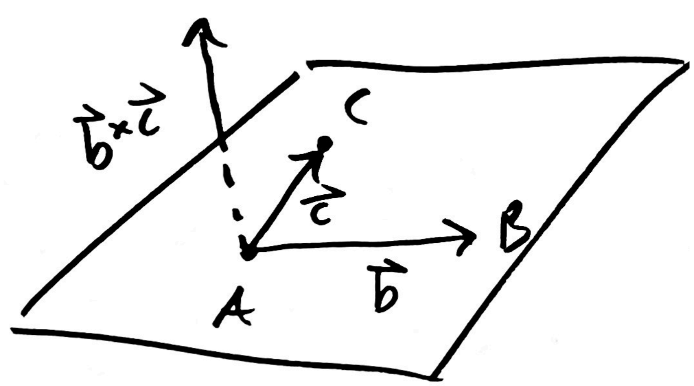
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & 0 & 3 \end{vmatrix} = \left[\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}, \underbrace{\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}}_{1 \cdot 0 - 2 \cdot 2} \right]$$

$$= [6, -5, -4]$$

Planet er gitt ved $\underline{6x - 5y - 4z = 0}$

Finn likningen til planet gjennom

9 $A(1,2,3)$ $B(1,2,1)$ $C(1,1,1)$



$$\vec{b} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [1,2,1] - [1,2,3] = [0,0,-2].$$

$$\vec{c} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = [1,1,1] - [1,2,3] = [0,-1,-2].$$

vektoren $\vec{n} = [1,0,0]$ står vinkelrett på både $\vec{b}$ og $\vec{c}$.

$$[1,0,0]\cdot[x,y,z] = d$$

$$x = d$$ setter inn punkt A (eller B, C)

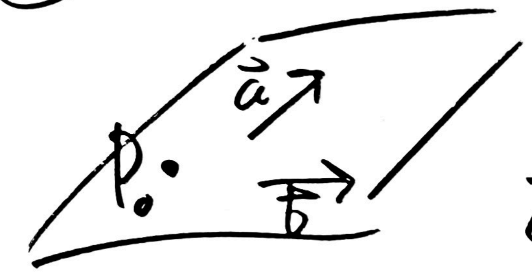
$$d = 1$$

så planet er gitt ved $\underline{x = 1}$

(10)

Finn likningen til planet som innholder $P_0(1,1,2)$ og som er utspent av vektorene $\vec{a} = [10, -15, 20]$

$\vec{b} = [28, -7, -14]$.



$\vec{a} = 5[2, -3, 4]$

$\vec{b} = 7[4, -1, -2]$

$$\vec{a} \times \vec{b} = 5 \cdot 7\,[2,-3,4] \times [4,-1,-2] = 35 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= 35\left[\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix}, -\begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix}, \begin{vmatrix} 2 & -3 \\ 4 & -1 \end{vmatrix} \right]$$

$$= 35[10, 20, 10] = 350[1, 2, 1]$$

velger $\vec{n} = [1, 2, 1]$

(11)

$$[x, y, z] \bullet [1, 2, 1] = d$$

$$x + 2y + z = d$$

setter inn koordinatene til $P_0(1, 1, 2)$

$$1 + 2 \cdot 1 + 1 = d$$

$$\underline{4 = d}$$

Likningen er $\underline{x + 2y + z = 4}$

$$(\Leftrightarrow 350x + 700y + 350z = 1400)$$

Finn en normalvektor til $2x - 4y + 6z = 5$

$[2, -4, 6] = 2[1, -2, 3]$. En normalvektor er $\underline{[1, -2, 3]}$.

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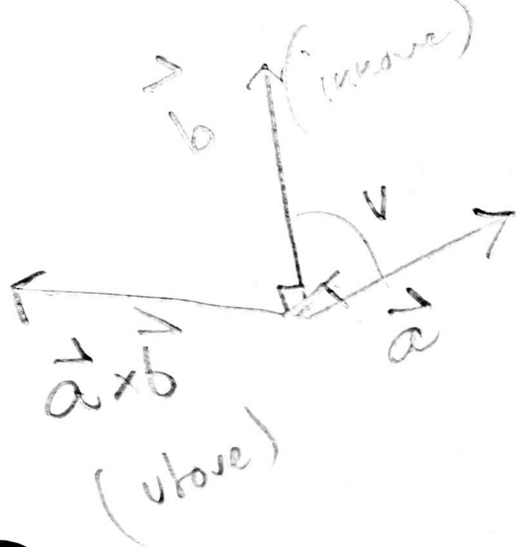
Finn et punkt i planet

$$2x - 4y + 6z = 5.$$

$y = 0, z = 0$ gir $2x = 5$ så $x = 5/2$

$(2.5, 0, 0)$ er et punkt i planet.

Øving



$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin v$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ & \vec{a} & \\ & \vec{b} & \end{vmatrix}$$

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$$[2, -4, 2] \times [-7, 3, 4]$$

$$= \begin{vmatrix} i & j & k \\ 2 & -4 & 2 \\ -7 & 3 & 4 \end{vmatrix} = \left[\begin{vmatrix} -4 & 2 \\ 3 & 4 \end{vmatrix}, -\begin{vmatrix} 2 & 2 \\ -7 & 4 \end{vmatrix}, \begin{vmatrix} 2 & -4 \\ -7 & 3 \end{vmatrix} \right]$$

$$= [-22, -22, -22]$$

$$= \underline{-22[1, 1, 1]}$$

14

Finn en likning for planet som inneholder $P(-2,3,-4)$ og som er utspent av $\vec{a} = [4,-4,12]$ og $\vec{b} = [6,-18,24]$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -4 & 12 \\ 6 & -18 & 24 \end{vmatrix} = 4\cdot 6 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= 4\cdot 6 \left(\begin{vmatrix} -1 & 3 \\ -3 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} \vec{k} \right)$$

$$= 4\cdot 6 \underline{[5, -1, -2]}$$

velger normalvektoren $\vec{n} = [5,-1,-2]$

Planet $[x,y,z]\cdot\underbrace{[5,-1,-2]}_{\vec{n}} = 5x - y - 2z = d$

setter inn koord. til $P_0(-2,3,-4)$

$5(-2) - (3) - 2(-4) = d$

$-10 - 3 + 8 = -5$

$$5x - y - 2z = -5$$

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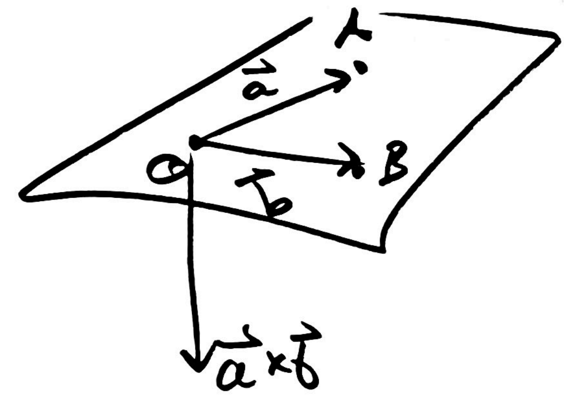
Finn likningen til planet gjennom punktene O $A(1,1,1)$ $B(1\ 2\ 3)$



$\vec{a} = \vec{OA} = [1,1,1]$

$\vec{b} = \vec{OB} = [1,2,3]$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

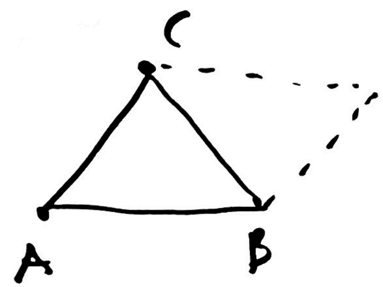
$$= \left[\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right]$$

$$\vec{a}\times\vec{b} = [1, -2, 1]$$

Likning for planet $x - 2y + z = 0$

14.154 $A(6,2,0)$ $B(3,6,0)$ $C(0,0,t)$

Bestem t slik at areal ΔABC blir 25.



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Areal $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$\vec{AB} = \vec{OB} - \vec{OA} = [3,6,0] - [6,2,0]$

$= [-3,4,0]$

$\vec{AC} = [0,0,t] - [6,2,0] = [-6,-2,t]$.

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 4 & 0 \\ -6 & -2 & t \end{vmatrix} = \left[\begin{vmatrix} 4 & 0 \\ -2 & t \end{vmatrix}, -\begin{vmatrix} -3 & 0 \\ -6 & t \end{vmatrix}, \begin{vmatrix} -3 & 4 \\ -6 & -2 \end{vmatrix} \right]$$

$= [4t, 3t, 30]$

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$2(\text{Areal } \Delta ABC) = |\vec{AB} \times \vec{AC}|$

$2 \cdot 25 = |[4t, 3t, 30]|$ kvadrere

$50 = \sqrt{(4t)^2 + (3t)^2 + 30^2}$

$2500 = \underbrace{16t^2 + 9t^2}_{25t^2} + 30^2$

$2500 = 25t^2 + 900$

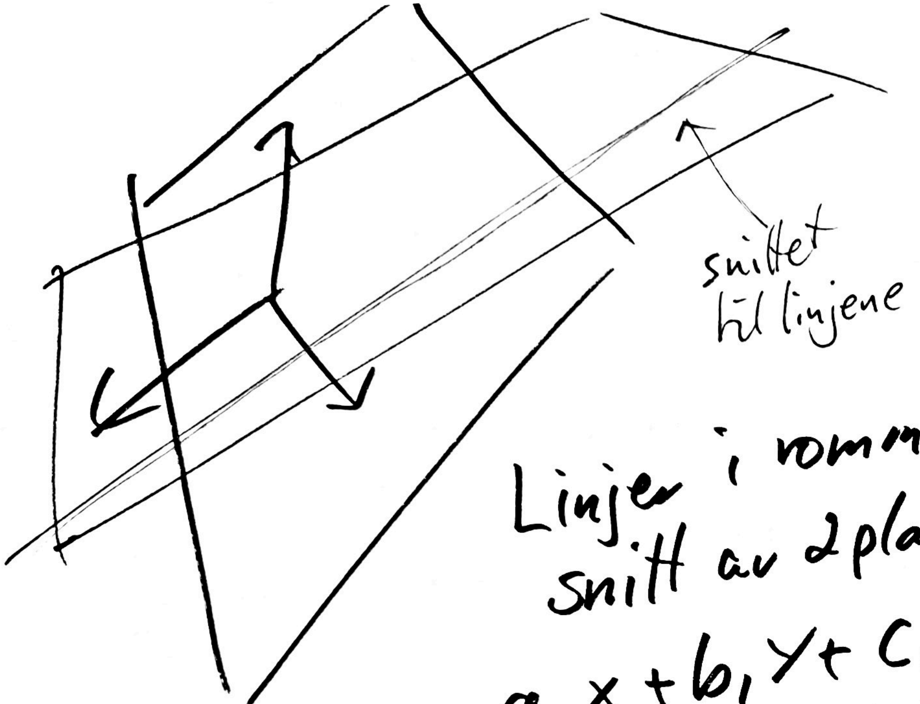
$1600 = 25 \cdot t^2$

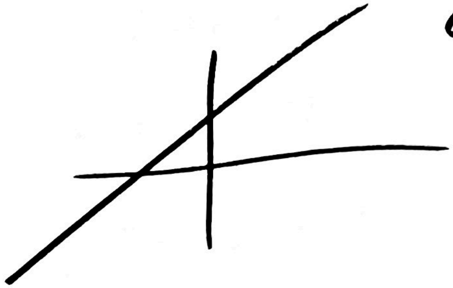
$\frac{1600}{25} \cdot \frac{4}{4} = t^2$

$\frac{6400}{100} = t^2$ så $t^2 = 64$

$t = \pm\sqrt{64}$

$t = \underline{\underline{\pm 8}}$



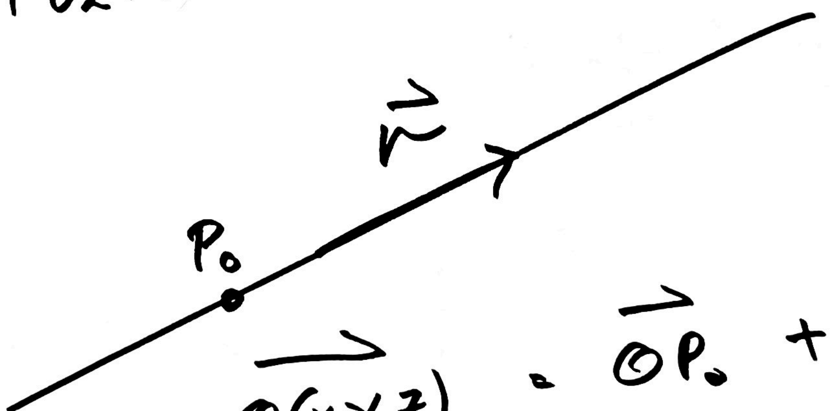


$ax+by=c$

Linjer i rommet
snitt av 2 plan:

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$

Parametrisering



$$\overrightarrow{O(x,y,z)} = \overrightarrow{OP_0} + t\vec{r}.$$

(19)

Parametiser linjen som er snittet av planene

$$x + y + 2z = 2$$
$$-2x + y + 3z = -1$$

1) Finne et punkt P_0 som ligger i begge planene.

2) Finn retningsvektor til snittlinjen.

1) sette $x = 0$:

$$y + 2z = 2$$
$$y + 3z = -1$$

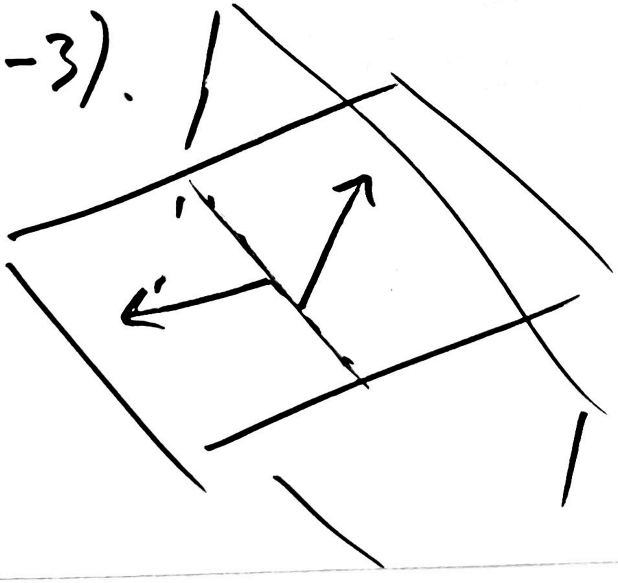
$-L1$ begges til $L2$

$$z = -1 - 2 = -3.$$
$$y + 2(-3) = 2$$

så $y = 2 + 6 = 8$

$P_0(0, 8, -3)$.

2)



Snittlinjen er normal til begge normal-vektorene!

En retningsvektor til snittlinjen er kryssproduktet av de to normalvektorene.

$$\vec{r} = [1,1,2] \times [-2,1,3]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -2 & 1 & 3 \end{vmatrix} = \left[\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} \right]$$

$$= [1, -7, 3]$$

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En parametrisering av linjen:

$$\underline{[x,y,z] = [0,8,-3] + t[1,-7,3]}$$