9.nov
2021

17.5-6 Geometriske rekker.

Følge $a_1, a_2, a_3, \ldots, a_n$

Rekke $a_1 + a_2 + a_3 + \cdots + a_n$

$= \sum_{i=1}^{n} a_i$ "sum av a_i fra $i=1$ til $i=n$"

Σ, σ sigma (gresk)

$$\sum_{i=3}^{8} i = 3+4+5+6+7+8$$

$$\sum_{i=1}^{100} i^2 = 1+4+9+16+\cdots+100^2$$

partall $2+4+6+\cdots+200 = \sum_{i=1}^{100} 2i$

(↑ stigende alltid 1)

Aritmetisk følge $\quad a_{n+1} = a_n + d \quad$ alle n

Geometriske følger $\quad a_{n+1} = k \cdot a_n \quad$ alle n

d differanse : $a_{n+1} - a_n$

k kvotient : a_{n+1} / a_n

$a_n \neq 0$.

Geometriske følger

$2, \frac{-1}{2}, \frac{1}{8}, \frac{-1}{32}, \ldots \quad k = \frac{-1}{4}$

$1, -1, 1, -1, 1, -1, \ldots \quad k = -1$

$4, 8, 16, 32, \ldots \quad k = 2$

Er $\quad \overset{a_1}{2}, \overset{a_2}{4}, \overset{a_3}{6}, \ldots \quad$ geometrisk.

$\frac{a_2}{a_1} = \frac{4}{2} = 2 \qquad \frac{a_3}{a_2} = \frac{6}{4} = \frac{3}{2}.$ ikke geom.

$a_1 = 3 \qquad a_4 = 24$

Finn geom. følger med disse egenskapene.

$a_4 = k a_3 = k^3 \cdot a_1$

$a_2 = k \cdot a_1$

$a_3 = k \cdot a_2 = k^2 a_1$

$24 = k^3 \cdot 3$ $\qquad (k^{n-1} \cdot a_1)$

$k^3 = \frac{24}{3} = 8 = 2^3$

$\underline{k = 2}$ $\qquad$ så $a_n = \underline{\underline{2^{n-1} \cdot 3}}$

Generelt $\qquad a_n = k^n a_0 = k^{n-1} a_1$

$= k^{n-m} a_m$

Finnes det en geom. følge med $a_4 = 5$ og $a_6 = -20$?

$a_6 = k^2 \cdot a_4$ så $-4 = k^2$

ingen reell løsning.

oppg. Finn alle geom. følger $a_4 = 5$ og $a_6 = 20$

Finn a_n.

$$\frac{a_6}{a_4} = \frac{20}{5} = 4 = k^2.$$

$k = -2$ og $k = 2$

$k=2$ $\quad a_n = 2^{n-4} \cdot 5 = \frac{5}{16} \cdot 2^n.$

$$\frac{5}{8}, \frac{5}{4}, \frac{5}{2}, \underset{a_4}{5}, 10, 20, 40, \ldots$$

$k=-2$ $\quad a_n = (-2)^{n-4} \cdot 5 = \frac{5}{16}(-2)^n$

$$\frac{-5}{8}, \frac{5}{4}, \frac{-5}{2}, 5, -10, 20, -40, \ldots$$

Geometrisk rekke

$$\underbrace{1+k+k^2+\cdots+k^{n-1}}_{n \text{ ledd}} = \begin{cases} \frac{1-k^n}{1-k} & k \neq 1 \\ n & k=1 \end{cases}$$

$k=2$ $$1+2+4+8+\cdots+2^{n-1} = \frac{1-2^n}{1-2} = 2^n-1$$

$k=\frac{1}{2}$ $$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}} = \frac{1-(1/2)^n}{1-1/2} = 2\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$\frac{x^n-1}{x-1}\left(=\frac{1-x^n}{1-x}\right) = 1+x+x^2+\cdots+x^{n-1}.$$

$n=3$ $$x^3-1 = (1+x+x^2)(x-1)$$

bevis for summen til en geometrisk rekke

$k = 1$ ✓

$k \neq 1$

$$S_n = 1 + k + k^2 + \cdots + k^{n-1}$$

$$k \cdot S_n = \quad k + k^2 + \cdots + k^{n-1} + k^n$$

(tar differansen)

$$S_n - k \cdot S_n = 1 - k^n$$

$$(1-k) S_n = 1 - k^n$$

så $$S_n = \frac{1-k^n}{1-k}.$$

$$k^m + k^{m+1} + \cdots + k^{n-1} = k^m (1 + k + k^2 + \cdots + k^{n-m-1})$$

$$= k^m \cdot \frac{1-k^{n-m}}{1-k}$$

$$= \frac{k^m - k^n}{1-k}$$

$k \neq 1$

eks 2012 des.

Geom. rekke

$$2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\cdots+\frac{128}{729}=\frac{2^7}{3^6}$$

$$\left(\begin{aligned}&81\cdot 9\\&=810-81\\&=729\end{aligned}\right.$$

Finn kvotienten. $k=\frac{8/9}{4/3}=\underline{\frac{2}{3}}$

Finn summen.

$$2(1+k+k^2+\cdots+k^6)$$

$$=2\cdot\frac{1-k^7}{1-k}=2\,\frac{1-(2/3)^7}{1-2/3}=2\cdot 3\left(1-\left(\frac{2}{3}\right)^7\right)$$

$$=2\left(3-\frac{2^7}{3^6}\right)=\underline{2\left(3-\frac{128}{729}\right)}$$

Oppg. Finn summen av tall på forme $2^i\cdot 5^j$ $\quad 0\le i\le 10$, $0\le j\le 5$

$1, 2, 4, 5, 8, 10, 16, 20, \ldots$

$$(1+2^1+2^2+\cdots 2^{10})(1+5^1+\cdots+5^5)$$

$$\left(\frac{1-2^{11}}{1-2}\right)\left(\frac{1-5^6}{1-5}\right) = \underline{(2^{11}-1)\cdot\frac{1}{4}(5^6-1)}$$

$$= \underline{7\,995\,582}$$

oppg.

En geometrisk rekke har første ledd lik 5 og kvotient $k = 1/3$. Hva er summen av de n første leddene.

$$\underbrace{5 + 5\cdot\frac{1}{3} + 5\cdot\left(\frac{1}{3}\right)^2 + \cdots + 5\left(\frac{1}{3}\right)^{n-1}}_{n \text{ ledd}}$$

$$= 5\Big(\underbrace{1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^{n-1}}_{\frac{1-(1/3)^n}{1-1/3}}\Big) = \frac{5}{2/3}\left(1-\left(\frac{1}{3}\right)^n\right)$$

$$= \underline{\frac{15}{2}\left(1-\left(\frac{1}{3}\right)^n\right)}$$

$$\underbrace{1 - 1 + 1 - 1 + 1 - \cdots + (-1)^{n-1}}_{n \text{ ledd}} \qquad k = -1$$

$S_1 = 1$

$S_2 = 0$

$S_3 = 1$

$$S_n = \begin{cases} 1 & n \text{ oddetall} \\ 0 & n \text{ jevn} \end{cases}$$

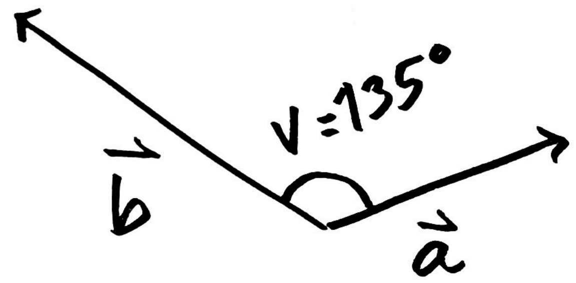
$$\sum_{i=0}^{n-1} (-1)^i = \frac{1-(-1)^n}{1-(-1)}$$

$$= \frac{1-(-1)^n}{2} = \begin{cases} 1 & n \text{ odde} \\ 0 & n \text{ jevn.} \end{cases}$$

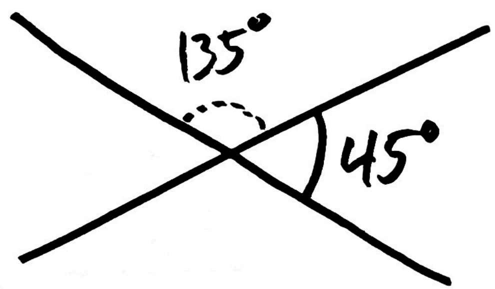
oblig3

oppg 1

Øving.



Lignende eksempel.



vinkel mellom to vektorer $\in [0, 180°]$

linjer $\in [0, 90°]$

$$1+k+\cdots+k^{n-1} = \sum_{i=0}^{n-1} k^i = \begin{cases} \dfrac{1-k^n}{1-k} & k \neq 1 \\ n & k = 1. \end{cases}$$

$$S = \frac{1}{3} - \frac{2}{3} + \frac{4}{3} - \frac{8}{3} - + \cdots + \frac{\overbrace{1024}^{2^{10}}}{3}$$

kvotient $k = -2$.

Finn summen S.

$$S = \frac{1}{3}\left(1 - 2 + 4 - 8 + \cdots + (-2)^{10}\right)$$

$$= \frac{1}{3}\left(\frac{1-(-2)^{11}}{1-(-2)}\right) = \frac{1}{3^2}\left(1-(-2)^{11}\right)$$

$$= \frac{1}{3^2}\left(1-(-2048)\right) = \frac{2049}{9}$$

$$= \underline{227 + \frac{2}{3}}$$

Fibonacci følgen :

$$F_0 = 0 \qquad F_1 = 1$$

$$F_{n+2} = F_{n+1} + F_n \qquad n \geq 0$$

$$F_2 = F_0 + F_1 = 0 + 1 = 1 \qquad F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

Forholdet $\frac{F_{n+1}}{F_n} \rightsquigarrow \varphi = \frac{1+\sqrt{5}}{2} \hat{=} 1.618$ "gylne forhold"

(sjekk wikipedia "golden ratio")

Det finnes en formel for F_n

$$F_n = \frac{1}{\sqrt{5}} \left[\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^n}_{\varphi^n} - \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}_{\left(\frac{-1}{\varphi}\right)^n} \right]$$

sjekk notat om Fibonacci tall på hjemmesiden til kurset.

Fibonacci følgen $F_0 = 0, \quad F_1 = 1$

$F_{n+2} = F_{n+1} + F_n \qquad n \geq 0.$

geometriske rekker slik at $F_{n+2} = F_{n+1} + F_n$

$a_n = a_0 k^n$ setter inn

$k^{n+2} = k^{n+1} + k^n \quad / k^n$

$k^2 = k + 1$

2.gradspol. $k^2 - k - 1 = 0$

$k = \dfrac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

$k_1 = \dfrac{1+\sqrt{5}}{2} \approx 1.618$ "gylne snitt"

$k_2 = \dfrac{1-\sqrt{5}}{2} = \dfrac{-1}{k_1} = -0.618$

$(k - 1 = \frac{1}{k})$

$$a_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

oppfyller $a_{n+2} = a_{n+1} + a_n \quad n \geq 0.$

bestemme a og b slik at $F_0 = a_0 = 0$

$F_1 = a_1 = 1$

$a_0 = a + b = 0 \qquad a = -b$

$a_1 = a\left(\frac{1+\sqrt{5}}{2}\right) + (\underbrace{-a}_{b})\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$a\left(\frac{1+\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2}\right) = \sqrt{5} \cdot a = 1$

$a_0 = F_0,\ a_1 = F_1$ samme rekursiv beskrivelse

$a = \frac{1}{\sqrt{5}} = -b$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$a_n = F_n$ for alle n.

$$\frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2}$$ når n blir stor.

Finn tallene T_n slik at $T_0 = 3$ $T_1 = 4$

$$T_{n+2} = T_{n+1} + T_n$$

bestemme a og b slik at $a_0 = 3$ og $a_1 = 4$

for da $a_n = T_n$ for alle n.

$$a_0 = a + b = 3$$

$$a_1 = a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 4$$

$$= a\varphi + b\left(\frac{-1}{\varphi}\right) = 4$$

$$b = 3 - a$$

$$a\varphi + (3-a)\left(\frac{-1}{\varphi}\right) = 4$$

$$a\left(\varphi + \frac{1}{\varphi}\right) = 4 + \frac{3}{\varphi}.$$

$$a = \frac{4 + 3/\varphi}{\varphi + 1/\varphi} = \frac{4\varphi + 3}{\varphi^2 + 1}$$

$$T_n = \left(\frac{4\varphi+3}{\varphi^2+1}\right)\varphi^n + \left(3 - \frac{4\varphi+3}{\varphi^2+1}\right)\left(\frac{-1}{\varphi}\right)^n$$