Fauske

17.7. Uendelige rekker og følger.

10. nov
2021

Uendelig følge $a_1, a_2, a_3, \ldots$

Eksempler 1) $a_n = \frac{n}{n+1}$ $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

2) $a_n = -1+2n$ $1, 3, 5, 7, \ldots$

3) $a_n = x^n$, $x=2$, $2, 4, 8, 16, \ldots$

$x = \frac{1}{2}$ $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$

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4) $a_n = \sin(n^\circ)$ $\sin(1^\circ), \sin(2^\circ), \ldots, \sin(360^\circ)$,

$\sin(361^\circ), \ldots$

$a_1, a_2, a_3, \ldots$ <u>konvergerer</u> til a

hvis a_n nærmer seg a når n

blir stor. Vi skriver da $\lim_{n\to\infty} a_n = a$.

(lim = grense)

Eksemplene

1) $a_n = \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} a_n = 1$. følgen konvergerer til 1.

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2) $a_n = 2n-1$ $\underbrace{\text{konvergerer ikkje.}}_{\text{divergere}}$

3) $a_n = x^n$

$|x| < 1$ konvergere til 0.

$x > 1$ divergere ($x^n \to \infty$ når $n \to \infty$)

$x=1$ $\quad$ $x^n=1$ for alle n $\quad$ $1, 1, 1, 1, \dots$

konvergere til 1

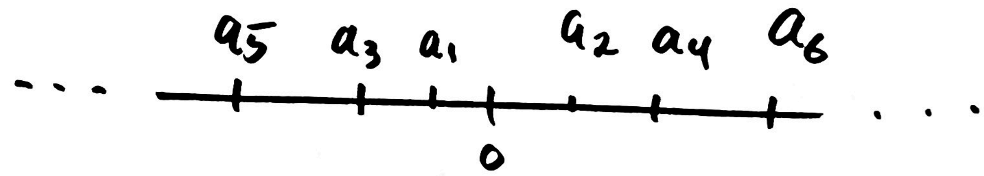
$x=-1$ $\quad$ $-1, 1, -1, 1, -1, 1, \dots$

divergere.

$x<-1$ $\quad$ $|x^n| \to \infty$

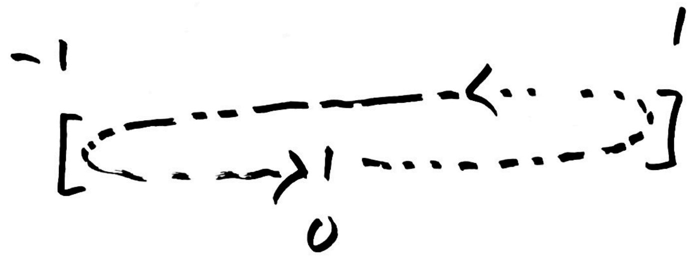
fortegnet til x^n alternera

3



divergerer.

4) $\sin(n^\circ)$



divergerer.

Presis betydning av $\lim_{n\to\infty} a_n = a$

Dette vil si: For enhver $\epsilon > 0$
så finnes det en N
slik at for $n \geq N$ så er
$|a_n - a| < \epsilon$.

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Oppg. Konvergerer $a_n = \dfrac{3n^2-1}{(2n+1)(n-1)}$, $n \geq 2$
i så fall hva er grensen?

Pol. div $\dfrac{3n^2-1}{(2n+1)(n-1)} = \dfrac{3}{2} + \underbrace{\dfrac{ax+b}{(2n+1)(n-1)}}_{\text{rest}} \to 0$ når $n \to \infty$

Konvergerer til $\underline{3/2}$.

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Uendelige rekker

$a_1 + a_2 + a_3 + \cdots$ konverger og har sum S

hvis følgen av <u>delsummer</u>

$$S_n = a_1 + a_2 + \cdots + a_n$$

konvergerer og har grense S.

($S_1, S_2, S_3, \ldots$ følgen av delsummer)

$1 + x + x^2 + \cdots$

$$S_n = 1 + x + x^2 + \cdots x^{n-1} = \begin{cases} \frac{1 - x^n}{1 - x} & x \neq 1 \\ n & x = 1 \end{cases}$$

$|x| \geq 1$ divergerer.

$|x| < 1$ konvergerer til $\frac{1}{1-x}$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \frac{1}{1-1/2} = 2.$$

$$1+0.99+(0.99)^2+\cdots = \frac{1}{1-0.99} = \frac{1}{0.01} = \underline{100}$$

$$1.001001001\ldots = 1+\frac{1}{1000}+\frac{1}{(1000)^2}+\frac{1}{(1000)^3}+\cdots$$

$$= \frac{1}{1-\frac{1}{1000}} = \frac{1}{0.999}\cdot\frac{1000}{1000} = \underline{\frac{1000}{999}}$$

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$$1.1111 = \sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = \frac{1}{1-1/10} = \frac{1}{0.9} = \frac{10}{9}$$

Sà $0.1111\ldots = \frac{1}{9}$

$$0.232323\ldots = 0.\underline{23} = 23(0.01010101\ldots)$$

$$= 23\left(\frac{1}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n\right) = \frac{23}{100}\cdot\frac{1}{1-0.01}$$

$$= \frac{23}{100}\cdot\frac{1}{0.99} = \underline{\frac{23}{99}}$$

Oppg Vis at $0.9999\ldots = 0.\underline{9} = 1.$

$= 9 \cdot 0.11111\ldots = 9 \cdot \frac{1}{9} = 1.$

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$\frac{1}{7}$ som desimaltall

$$\frac{1}{7} = \left(\frac{10}{7}\right)0.1 = \left(1+\frac{3}{7}\right)0.1 = 0.1 + \left(\frac{30}{7}\right)0.01$$

$$= 0.1 + \left(4 + \frac{2}{7}\right)0.01 = 0.14 + \underbrace{\left(\frac{20}{7}\right)}_{2+\frac{6}{7}}0.001 = 0.142 + \underbrace{\left(\frac{60}{7}\right)}_{\frac{56+4}{7}}10^{-4}$$

$$= 0.1428 + \underbrace{\left(\frac{40}{7}\right)}_{\frac{35+5}{7}}10^{-5} = 0.14285 + \underbrace{\left(\frac{50}{7}\right)}_{\left(\frac{49+1}{7}\right)} \cdot 10^{-6}$$

$$= 0.142857\,142857\ldots = 0.\underline{142857} \text{ periodisk}$$

$$0,d_1d_2d_3\dots = \sum_{i=1}^{\infty} \frac{d_i}{10^i}$$

desimaltall som en uendelig rekke.

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$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \underline{1 - \frac{1}{m+1}}$$

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right)$$

$$\lim_{m\to\infty} S_m = 1.$$

Så $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

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$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 + 1 = 2.$$

Så $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergerer og har sum mellom 1 og 2.

Summen er $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

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Hvis $\sum_{n=1}^{\infty} a_n$ konvergerer,
da må $a_n \to 0$ når $n \to \infty$

$S_n - S_{n-1} = a_n$ må gå mot 0
når $S_n \to S$.

Selv om $a_n \to 0$ når $n \to \infty$ behøver ikkje
rekken konvergere.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{1}{\sqrt{5}} + \cdots$$

$$S_m = \sum_{n=1}^{m} \frac{1}{\sqrt{n}} \geq \underset{\text{antall elementer}}{m} \cdot \underset{\text{minste element}}{\frac{1}{\sqrt{m}}} = \sqrt{m}$$

siden $S_m \geq \sqrt{m}$ divergerer rekken.

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Harmonisk rekke

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

divergerer.

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^s} \text{ konvergerer} \quad s > 1 \right)$$

$$\sum_{n=1}^{2^m} \frac{1}{n}$$

$$[\underbrace{2^k+1, \; 2^{\overbrace{k+1}^{2^k+2^k}}}_{2^k \text{ heltall}}]$$

$$1 \geq 2^k \cdot \underbrace{\frac{1}{2^k+1}}_{\text{største verdi}} \geq \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n} \geq 2^k \cdot \underbrace{\frac{1}{2^{k+1}}}_{\text{minste verdi}} = \frac{1}{2}.$$

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$$\sum_{n=1}^{2^m} \frac{1}{n} = \sum_{i=2^{m-1}+1}^{2^m} \frac{1}{n} + \sum_{i=2^{m-2}+1}^{2^{m-1}} \frac{1}{n} + \cdots$$

$$= \underbrace{\sum_{i=2^0+1}^{2^1} \frac{1}{n}}_{1/2} + \dots \frac{1}{1}$$

$$\geq 1 + m \cdot \frac{1}{2}.$$

$$1+m \geq \sum_{n=1}^{2^m} \frac{1}{n} \geq 1 + \frac{m}{2}$$

Dette viser at $\sum_{n=1}^{\infty} \frac{1}{n}$ divergerer.

$2^{10} = 1024$ så $2^{30} > 10^9$

$$\sum_{n=1}^{10^9} \frac{1}{n} < 1 + 30 = 31.$$