15.11
2021

17.8 Potensrekker og geometriske rekker med variabel kvotient

1)

Geometrisk rekke $a_0 + a_0 k + a_0 k^2 + a_0 k^3 + \cdots$

$a_0 (1 + k + k^2 + \cdots)$

$$\underbrace{1 + k + k^2 + \cdots k^{n-1}}_{n \text{ ledd}} = \begin{cases} \dfrac{1-k^n}{1-k} & k \neq 1 \\ n & k = 1 \end{cases}$$

Uendelig rekke $1 + k + k^2 + \cdots = \dfrac{1}{1-k}$ $|k| < 1$ konvergerer

$f(x) = 1 + x + x^2 + \cdots = \dfrac{1}{1-x}$ $|x| < 1$

funksjon x variabel.

La $\frac{1}{1-x} = \frac{1}{y}$ $\qquad y = 1-x$

$x = 1-y$

2 $\sum_{n=0}^{\infty} (1-y)^n = \frac{1}{y}$ $\qquad 0 < y < 2$

$1 + (1-y) + (1-y)^2 + \cdots$

$\frac{1}{0.98} = \frac{1}{1-0.02} = 1 + (0.02) + (0.02)^2 + (0.02)^3 + \cdots$

$1.020408\ldots$

$1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$ $\qquad |x| < 1$

Minner deg på Σ- notasjonen.

(3) Σ sum

til og med

$$\sum_{i=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

Fra og med. <u>steg på 1</u>

$$a_0 + a_2 + a_4 + \cdots = \sum_{i=0}^{\infty} a_{2i}$$

<u>Potensrekker</u> $r(x) = \sum_{i=0}^{\infty} a_i x^i$, $S_n = \sum_{i=0}^{n} a_i x^i$ polynom av grad n

delsum

Mange funksjoner kan beskrives som en potensrekke.

x radianer

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}$$

gyldig for alle x

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$$

Alle potensrekker $r(x)$ har en konvergensradius R $[0, \infty]$.

(4) $r(x)$ konvergerer for $|x| < R$
divergerer for $|x| > R$

$|x| = R$?

$\sum_{i=0}^{\infty} x^i$ konvergensradius 1
divergerer for $x = \pm 1$.

$r(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^2}$ konvergensradius 1

$r(1) = \sum_{i=1}^{\infty} \frac{1}{i^2}$ konvergerer $(\pi^2/6)$

$r(-1) = \sum_{i=1}^{\infty} \frac{(-1)^i}{i^2}$ — " —

5

$r(x) = \sum_{i=1}^{\infty} \frac{x^i}{i}$ konvergensradius 1

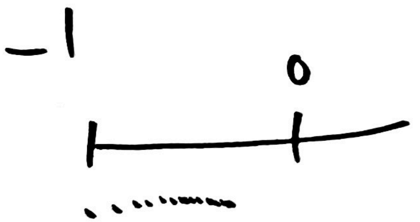
$x = 1$ $\quad r(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ harmonisk rekke.

divergerer.

$x = -1$ $\quad r(-1) = \underbrace{-1 + \frac{1}{2}} \underbrace{- \frac{1}{3} + \frac{1}{4}} \underbrace{- \frac{1}{5} + \cdots}$

$r(-1) < 0$

$r(-1) = -1 + \underset{+}{\left(\frac{1}{2} - \frac{1}{3}\right)} + \underset{+}{\left(\frac{1}{4} - \frac{1}{5}\right)} + \underset{+}{\cdots}$



$\underline{r(-1) \text{ konvergere}}$.

$\sum_{i=2}^{\infty} \left(\frac{x}{3}\right)^i$?

oppg. 1) Når konvergerer $\quad \sum_{i=2}^{\infty} (\sin x)^i$?

2) Hvor konvergerer

3) For hvilke x konvergere $\quad \sum_{i=1}^{\infty} \frac{x^{2i}}{\sqrt{i}}$?

6

1) $y = \frac{x}{3}$ $\sum_{i=2}^{\infty} y^i$ konv. for $|y|<1$, div for $|y|>1$

$|y| = \frac{1}{3}|x|$

$\frac{1}{3}|x| < 1$

$|x| < 3$ konv.

$|x| > 3$ div

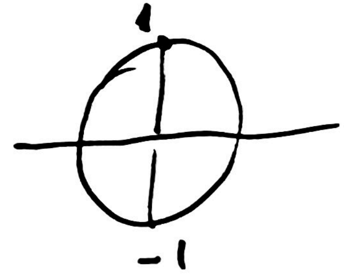
Konvergensradius er derfor $\underline{3}$

Rekken konv. $\underline{-3 < x < 3}$.

2) $\sum_{i=2}^{\infty} (\sin x)^i$

konv. $|\sin x| < 1$

Alle $x \neq \frac{\pi}{2} + \pi \cdot n$ $n \in \mathbb{Z}$.



3) $\sum_{i=1}^{\infty} \frac{x^{2i}}{\sqrt{i}}$ konv for $|x|<1$, div for $|x|>1$

$x = \pm 1$

$\sum_{i=1}^{\infty} \frac{((\pm 1)^2)^i}{\sqrt{i}} = \sum_{i=1}^{\infty} \frac{1}{\sqrt{i}}$ div.

Konverger for $\underline{x \in \langle -1, 1 \rangle}$

$$\sqrt{1-x} = 1 - \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{16} + \frac{5x^4}{128}$$

Tilnærming $\quad \sqrt{1-x} = 1 - \frac{x}{2}$

$$\left(1-\frac{x}{2}\right)^2 = 1 - x + \frac{x^2}{4}$$

7

nyttig. $\quad \sqrt{17} = \sqrt{16\left(1+\frac{1}{16}\right)} = 4\sqrt{1-\left(-\frac{1}{16}\right)}$

$$\sim 4\left(1 - \frac{1}{2}\left(-\frac{1}{16}\right)\right)$$

$$= 4 + \frac{4}{2\cdot 16} = 4 + \frac{1}{8}$$

$$\sqrt{17} \simeq 4 + \frac{1}{8} = 4.125$$

(nøyaktig verdi: 4.12310...)

Oppg. Finn summen av $\frac{1}{n}$ hvor $n \geq 1$ og (muligens) delelig med 7 eller 11 og ingen andre primtall.

⑧ $\sum\limits_{i,j \geq 0}^{\infty} \frac{1}{7^i 11^j}$ $\qquad 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{49} + \frac{1}{77} + \frac{1}{121} + \dots$

$$\left(\sum_{i=0}^{n} \frac{1}{7^i}\right)\left(\sum_{j=0}^{m} \frac{1}{11^j}\right) = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{1}{7^i 11^j}$$

$$\sum_{i=0}^{\infty} \left(\frac{1}{7}\right)^i = \frac{1}{1-1/7} = \frac{7}{6} \qquad \sum_{j=0}^{\infty} \left(\frac{1}{11}\right)^j = \frac{1}{1-1/11} =$$

$$\sum_{i,j \geq 0}^{\infty} \frac{1}{7^i 11^j} = \frac{7}{6} \cdot \frac{11}{10} = \frac{77}{60} = 1 + \frac{17}{60} \approx \underline{1.283}$$

$t = 1.2343434\ldots = 1.2\underline{34}$

Skriv tallet som en brøk.

$t = 1.2 + 0.034343434\ldots$

$= \frac{12}{10} + (0.1)\cdot(0.343434\ldots)$

$(34 \cdot 0.010101\ldots)$

9

$= \frac{12}{10} + \frac{1}{10} \cdot 34 \cdot \frac{1}{100}\Big(\underbrace{1 + \left(\frac{1}{100}\right) + \left(\frac{1}{100}\right)^2 + \cdots}_{\frac{1}{1-\frac{1}{100}}}\Big)$

$= \frac{12}{10} + \frac{34}{1000} \cdot \frac{100}{99} = \frac{12}{10} + \frac{34}{10} \cdot \frac{1}{99}$

$= \frac{1}{10}\left(12 + \frac{34}{99}\right) = \frac{1}{10} \cdot \frac{990 + 198 + 34}{99} = \frac{1222}{10 \cdot 99}$

$= \frac{1222}{990}$

Øving

Kan $1, 5, 9, 13, 19, 23, 27, \ldots$

være del av en aritmetisk følge?

10

Aritmetisk følge $a_{n+1} - a_n = d$ alle n.

$$a_n = a_k + (n-k) \cdot d$$

Ikke en aritmetisk rekke

$19 - 13 = 6$

men $13 - 9 = 9 - 5 = 5 - 1 = 4$

Finn summen av de n første positive oddetallene

slik at $k-1$ er delelig med 4.

$$a_n = 4n - 3$$

$1 + 5 + 9 + 13 + 17 + \cdots$

$$\sum_{i=1}^{n} (4n-3) = 4\left(\sum_{i=1}^{n} i\right) + (-3) \cdot n$$

(11)

$$= 4 \cdot \frac{n(n+1)}{2} - 3n$$

$$= 2n(n+1) - 3n$$

$$= 2n^2 + 2n - 3n$$

$$= \underline{2n^2 - n}$$

Alternativt:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right) = n\left(\frac{1 + 4n - 3}{2}\right)$$

$$= \underline{n(2n-1)}$$

oppg Hvor mange ledd må vi ha med i rekken ovenfor for at summen skal bli nærmest 20 000.

n stor $S_n = n \cdot (2n-1) \sim 2n^2$.

$S_n \sim 20\,000$

$2n^2 \sim 20\,000$

$n \sim 100$

$$S_{100} = 100(2\cdot 100 - 1) = 20\,000 - 100$$
$$= 19\,900$$

$$S_{101} = S_{100} + a_{101} = 20\,000 - 100 + \underbrace{4(101) - 3}_{401}$$
$$= \underline{20\,000 + 301}$$

(12)

S_{100} er nærmest 20 000

verdien er da lik 19900.

$$P_1 = P_0 + rP_0 = (1+r)P_0$$

r rente (p.a pro anno)

Renter

$$P_2 = (1+r)P_1 = (1+r)^2 P_0$$

$$\underline{P_n = (1+r)^n P_0}$$

n antall år.

13) Setter inn $P_0 = 1000$ kr årlig 1.1. fra 2000 til 2008.

Hvor mye penger har vi i starten av 2015

Setter inn P_0 9 ganger.

Penger 1.1.2008 :

$$P_0 + (1+r)P_0 + (1+r)^2 P_0 + \cdots (1+r)^8 P_0$$

$$= P_0\left(1 + \underbrace{(1+r) + (1+r)^2 + \cdots + (1+r)^8}_{\text{geometrisk rekke}}\right)$$

$$= P_0 \frac{1 - (1+r)^9}{1 - (1+r)} = \underline{\frac{P_0}{r}\left((1+r)^9 - 1\right)}$$

Pengene står 7 år i banken og pengemengden 1.1.2015 er da

$$\frac{P_0}{r}\left((1+r)^9 - 1\right) \cdot (1+r)^7$$

$$\frac{1000}{0.1}\left((1.1)^9 - 1\right)(1.1)^7 = \underline{26\,462,5 \text{ kr}}$$

14

Finn summen til rekken

(0613
131)

1) $\sum_{i=1}^{n} \frac{3^{2i+1}}{2^{4i}}$

2) $\sum_{i=1}^{n} 3^n$

1) $\sum_{i=1}^{n} 3 \frac{(3^2)^i}{(2^4)^i} = 3 \sum_{i=1}^{n} \left(\frac{9}{16}\right)^i$

$= 3 \cdot \frac{9}{16} \sum_{i=1}^{n} \left(\frac{9}{16}\right)^{i-1}$

$= 3 \cdot \frac{9}{16} \left(\frac{1-(9/16)^n}{1-9/16} \right)$

$= 3 \cdot \frac{9}{16} \cdot \frac{1-(9/16)^n}{7/16}$

$= \underline{\frac{27}{7}\left(1-\left(\frac{9}{16}\right)^n\right)}$

2) $\sum_{i=1}^{n} 3^n = \underbrace{3^n + 3^n + 3^n + \cdots + 3^n}_{n}$

15

$= n \cdot 3^n$

$k^a + k^{a+1} + \cdots + k^b$ $\quad a \leq b$ helltall

$k^a(1 + k + \cdots + k^{b-a})$

$k^a\left(\frac{1 - k^{b-a+1}}{1-k}\right) = \frac{k^a - k^{b+1}}{1-k}$ $\quad k \neq 1$