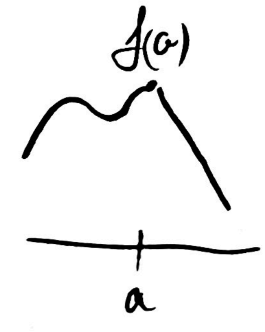
7 jan 2022 6.2 Kontinuitet 6.3 Asymptoter

$f(x)$ er kontinuerlig i $x=a$, hvis f er definert i a

(1) og $f(x) \to f(a)$ når $x \to a$.



$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

ikkje kontinuerlig kalles diskontinuerlig

Eks $f(x) = \begin{cases} \sin x & x<0 \\ \dfrac{x^3-1}{x-1} & 0 \le x < 1 \\ 3x & x \ge 1 \end{cases}$ funksjon gitt ved delt forskrift.

f kont for $x<0$, $0<x<1$, $x>1$ Hva skjer i $x=0$? og $x=1$.

2)

$x=0$ $\quad f(0) = \dfrac{0^3-1}{0-1} = 1.$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{x^3-1}{x-1} = \dfrac{-1}{-1} = 1.$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \sin x = 0$, så $\lim\limits_{x\to 0} f(x)$ eksisterer ikke.

$f(x)$ er diskontinuerlig i $x=0$ (hopp diskontinuitet)

$x=1$ $\quad f(1) = 3$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} 3x = 3.$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \dfrac{x^3-1}{x-1}$ type $\frac{0}{0}$ $= \lim\limits_{x\to 1^-} \dfrac{(x-1)(x^2+x+1)}{(x-1)}$

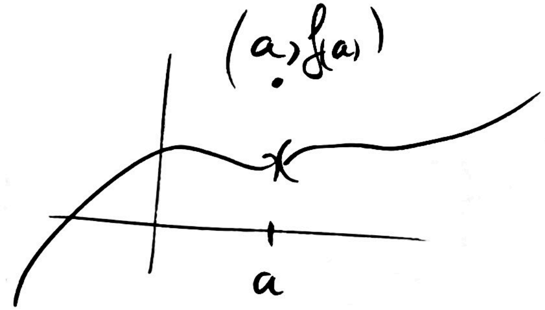
$= \lim\limits_{x\to 1^-} (x^2+x+1) = 1+1+1 = 3.$

$\lim\limits_{x\to 1} f(x) = 3 = f(1)$ så $f(x)$ er kont. i $x=1$.

$f(x)$ er kont. i alle punkt bortsett fra $x=-1$.

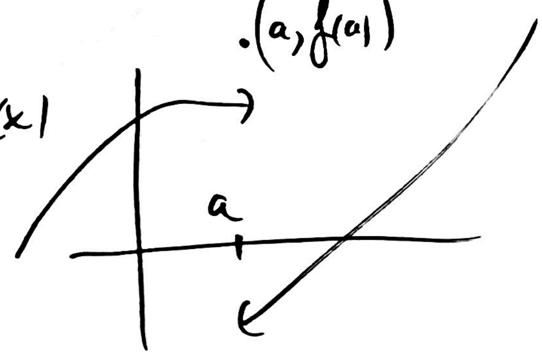
(3) Typer diskontinuitet

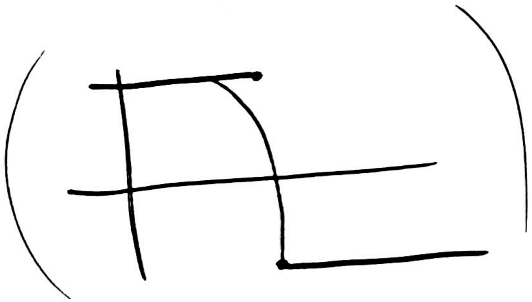
1) Hevbar diskontinuitet $\lim_{x\to a} f(x)$ eksistere $\neq f(a)$



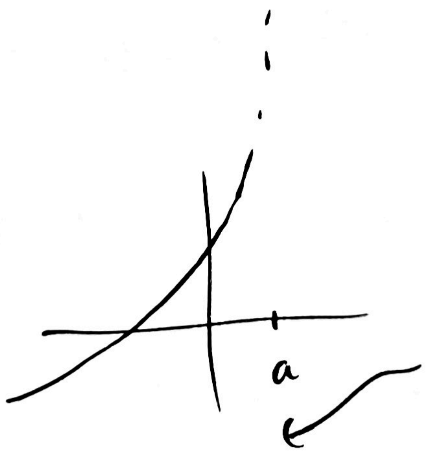
Kan gjøre f kont. i $x=a$ ved å endre $f(a)$ til $\lim_{x\to a} f(x)$.

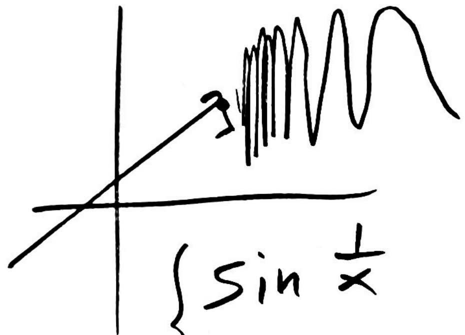
2) Hopp diskontinuitet $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$





3) Essensielle diskontinuiteter

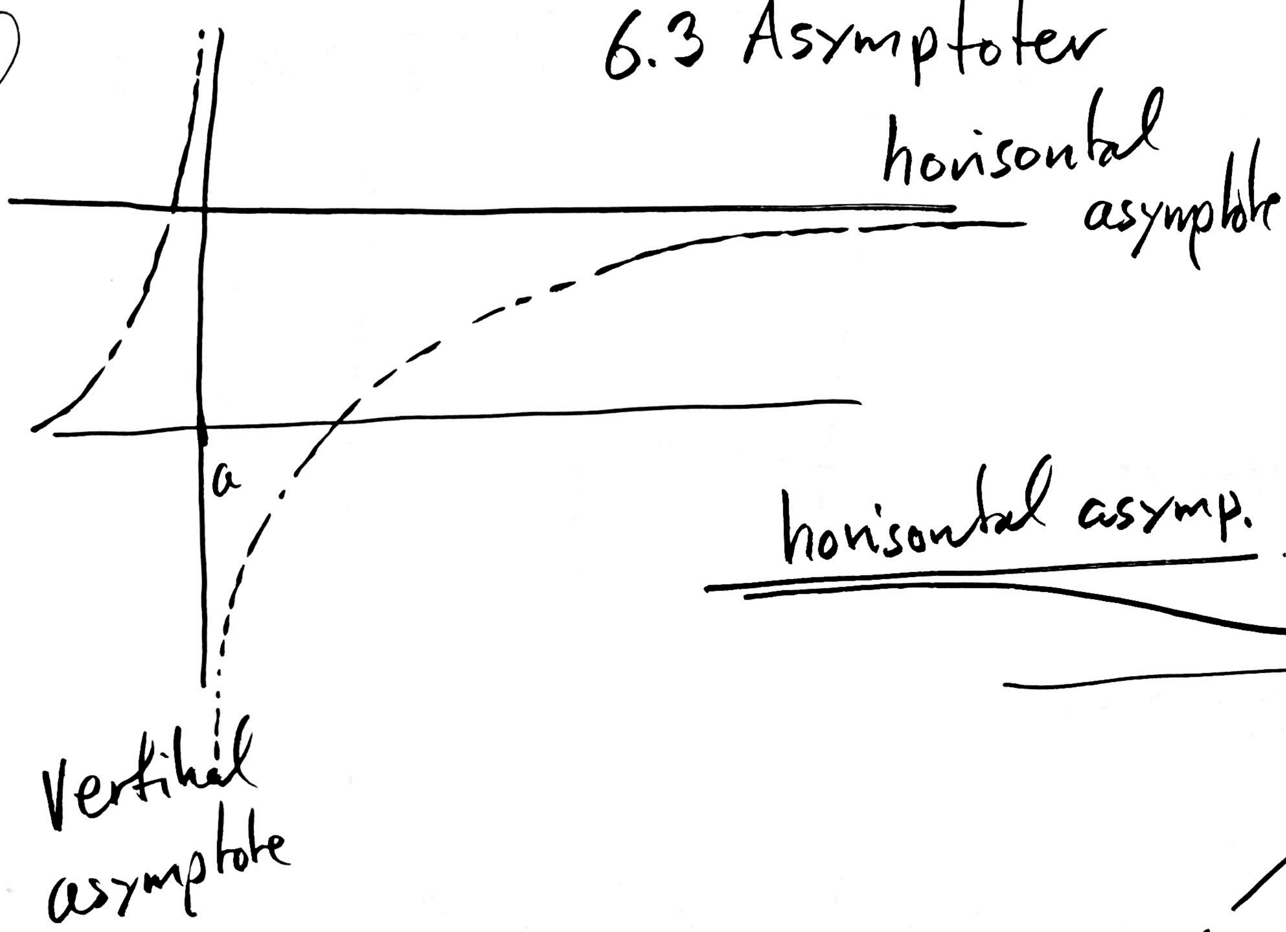


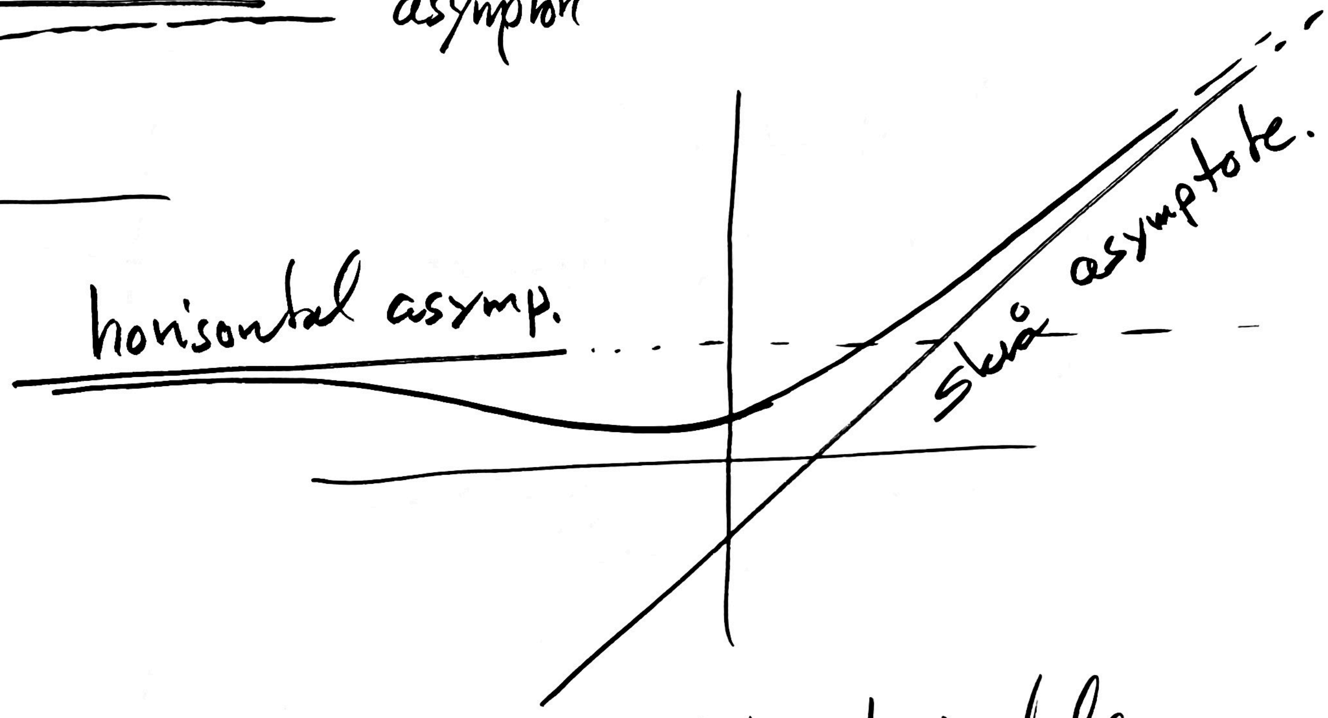


$$\begin{cases} \sin\frac{1}{x} & x>0 \\ 0 & x\le 0 \end{cases}$$

④

6.3 Asymptoter



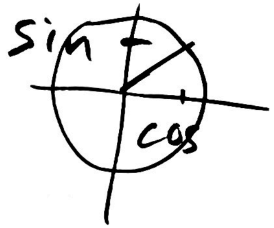


skrå asymptoter innholder horisontale asymptoter.

Det er 0, 1 eller 2 skrå (horisontale) asymptoter.

Det kan være vendelig mange Vertikale asymptoter.

$\tan x = \dfrac{\sin x}{\cos x}$ ikke def $x = \dfrac{\pi}{2} + \pi \cdot n$



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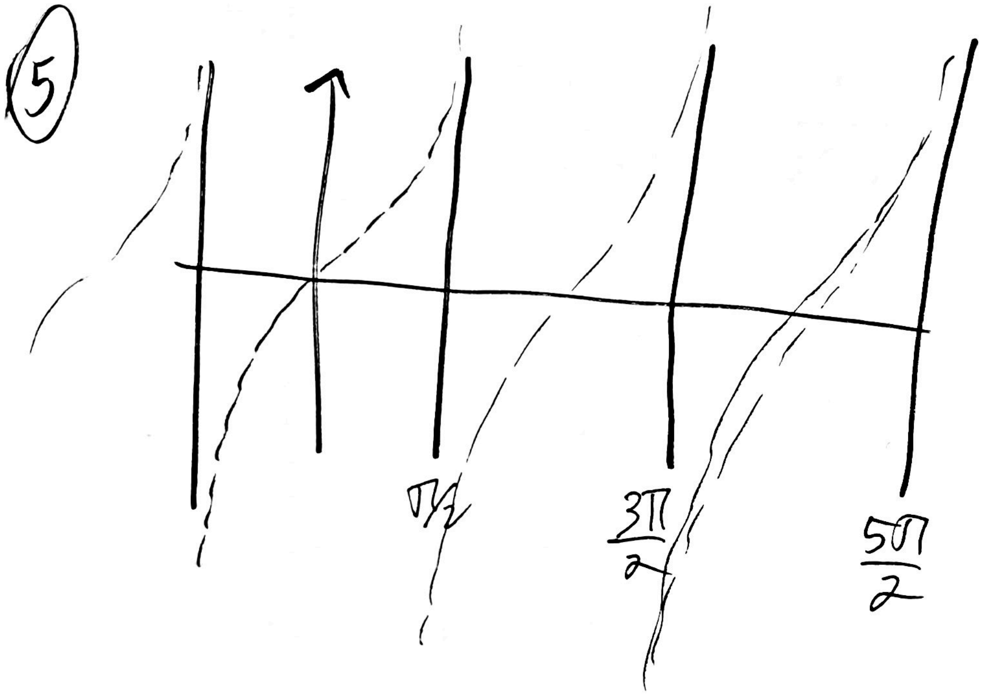


$\tan x$ har vertikale asymptoter i $x = \dfrac{\pi}{2} + \pi \cdot n$

$n \in \mathbb{Z}$

∞ mange vertikale asymptoter.

$f(x)$ har vertikal asymptote i $x = a$ hvis

$\lim\limits_{x \to a^-} f(x) = \infty$, $\lim\limits_{x \to a^-} f(x) = -\infty$

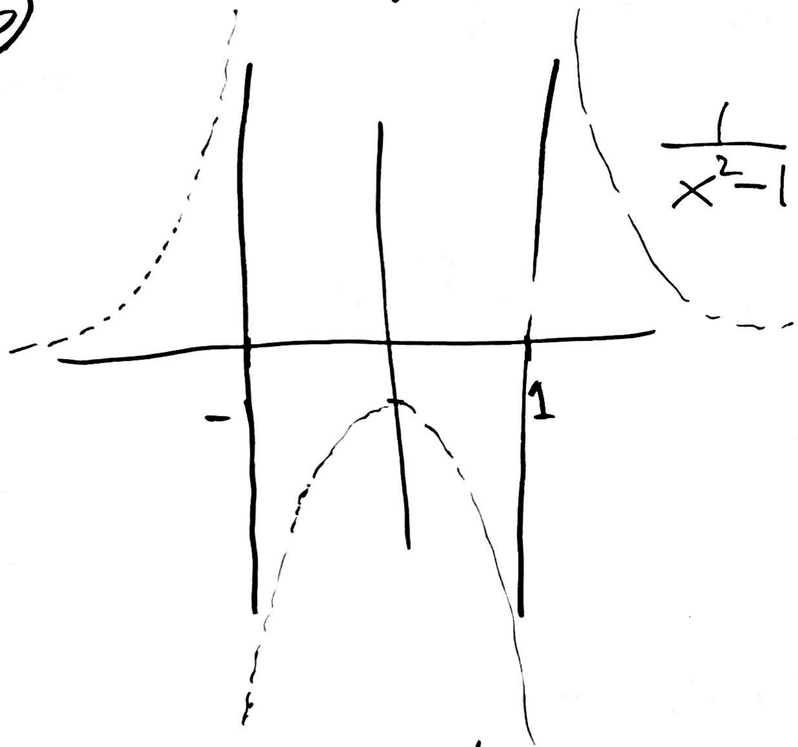
$\lim\limits_{x \to a^+} f(x) = \infty$, $\lim\limits_{x \to a^-} f(x) = -\infty$

$f(x)$ behøver ikke være def. i $x = a$.

⑥

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$$

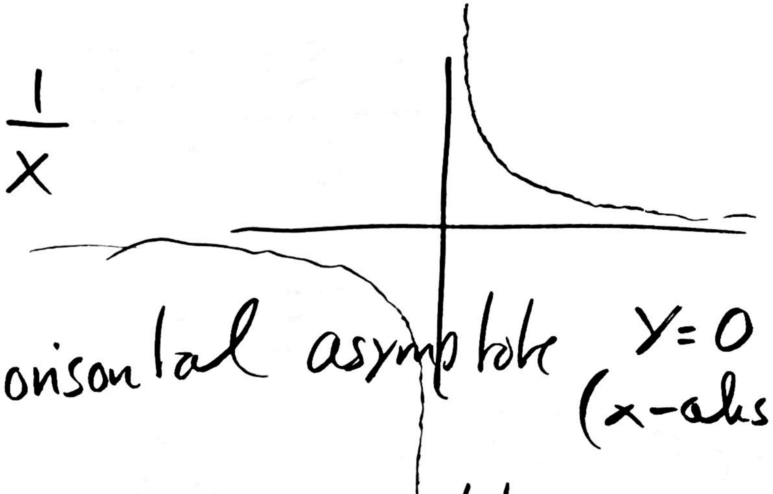
$$= \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$



Vertikal asymptoter $x=-1$
og $x=1$

horisontal asymptote $y=0$

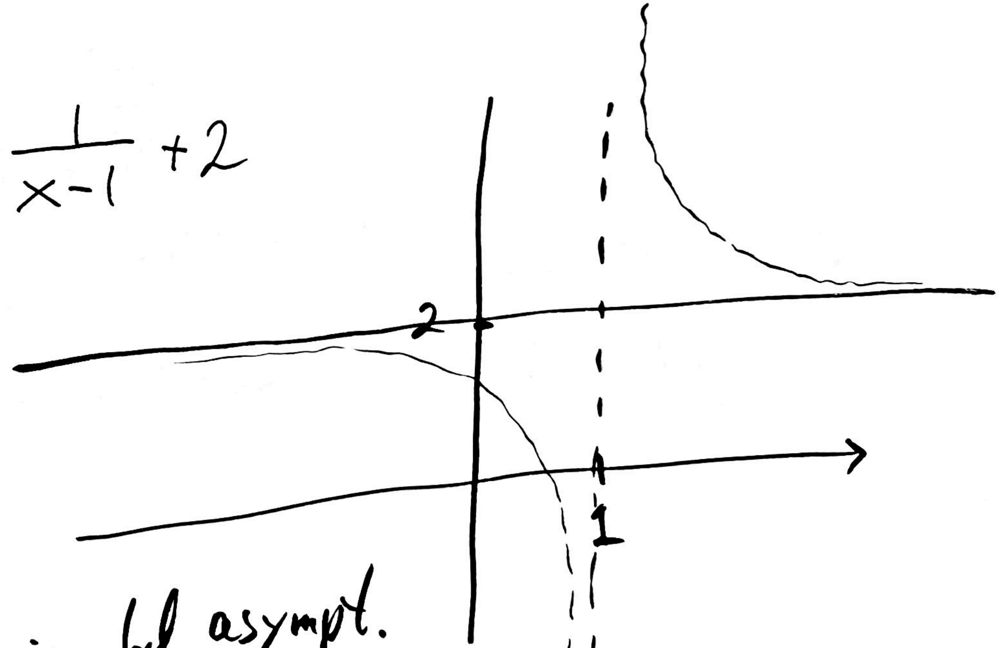
$\frac{1}{x}$



horisontal asymptote $y=0$
(x-aksen)

Vertikal asymptote $x=0$
(y-aksen)

$\frac{1}{x-1}+2$



horisontal asympt.
$y=2$

Vertikal asympt. $x=1$

Finn asymptoter til $r(x) = \dfrac{3x^2 + 3x + 1}{2x^2 - 3}$

$$\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \frac{3 + 3/x + 1/x^2}{2 - 3/x^2} = \frac{3}{2}$$

$y = 3/2$ horisontal asymptote.

nevneren lik 0 :

$$2x^2 - 3 = 0$$
$$x^2 = 3/2$$
$$x = \pm\sqrt{3/2} .$$

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telleren er ulik 0 for disse verdiene,

så vertikale asymptoter ; $x = -\sqrt{\tfrac{3}{2}}$

og $x = \sqrt{\tfrac{3}{2}}$.

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Asymptoter til $f(x) = \dfrac{x^2-1}{x^2-x} = \dfrac{(x+1)(x-1)}{x(x-1)} = \dfrac{x+1}{x}$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{1-1/x^2}{1-1/x} = 1$ horisontal asymptote $y = 1$

nevner lik 0 : $x^2 - x = x(x-1)$.

$x=0$ og $x=1$.

$\lim\limits_{x\to 0^+} f(x) = \infty$

$\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x+1}{x} = 2$

vertikal asymptote $x=0$ (y-aksen)

$y = ax + b$ er en skrå asymptote for $f(x)$

hvis $\lim_{x \to \infty} \left(f(x) - y(x) \right) = 0$

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eller $\lim_{x \to -\infty} \left(f(x) - y(x) \right) = 0$

$a = 0$: horisontal asymptote.

eks

$$g(x) = \begin{cases} 2 + \frac{1}{x} & x \leq -1 \\ \frac{1}{x+2} & x > -1 \end{cases}$$

ingen vertikale asymptoter.

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{1}{x+2} = 0$$

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} 2 + \frac{1}{x} = 2$$

$y = 0$ og $y = 2$ er begge horisontale asymptoter.

Rasjonale funksjoner $\quad r(x) = \frac{p(x)}{q(x)}$

Vertikale asymptoter $\quad$ bare i nullpunkt for $q(x)$
(Nullpunkt behøver ikkje være Vert. asymptoter)

(10)

\# V. asymptote $\leq$ deg. $q(x)$.

$$\frac{p(x)}{q(x)} = ax + b + \frac{s(x)}{q(x)} \qquad \deg s < \deg q$$

pol. div

skråasymptoter

deg p = deg $q + 1$ $\quad a \neq 0$ $\quad$ skrå asymp. $y = ax + b$

deg p = deg q $\quad a = 0,\ b \neq 0$ $\quad$ horisontal asymp $y = b$

deg p < deg q $\quad$ horisontal asymptote $y = 0$.

skrå asymptoter

$r(x) = 2x-3$ linje

sin egen skrå asymptot

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$f(x) = \frac{x^2+x-1}{x} = x+1-\frac{1}{x}$

skrå asymptote $x+1$.

$f(x)-(x+1) = \frac{-1}{x}$

så $\lim_{x\to\infty} f(x)-(x+1) = \lim_{x\to\infty} \frac{-1}{x} = 0$.

$\frac{x^2+2}{x-1} = x+1+\frac{3}{x-1}$

så $x+1$ er en skrå asymptote til $\frac{x^2+2}{x-1}$

$$
\begin{array}{l}
x^2+2 : x-1 = x+1+\frac{3}{x-1} \\
\underline{x^2-x} \\
x+2 \\
\underline{x-1} \\
3
\end{array}
$$

Finn asymptoter til $f(x) = \begin{cases} \dfrac{x-1}{x^3+1} & x < -1 \\ 1/3 & -1 \le x \le 1 \\ \dfrac{x}{x-1} & x > 1 \end{cases}$

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$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x-1}{x^3+1} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x-1} = \lim_{x\to\infty} \frac{1}{1-1/x} = 1$$

To horisontale asymptoter x-aksen og $y = 1$.

$x^3+1 = 0$ $\quad x = -1$

$$\begin{array}{l} x^3 + \quad +1 : x+1 = x^2 - x + 1 \\ \underline{x^3+x^2} \\ -x^2 \quad +1 \\ \underline{-x^2-x} \\ x+1 \end{array}$$

$x^2 - x + 1 > 0$ alle x

Løsning

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$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \cancel{\frac{x-1}{(x-1)(x^2-x+1)} = \lim_{x\to -1^-} \frac{1}{x^2-x+1}} = \frac{1}{3}$$

$$= \lim_{x\to -1^-} \frac{x-1}{(x+1)(x^2-x+1)} = \infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x}{x-1} = \infty$$

To vertikale asymptoter

$\underline{x=-1}$ og $\underline{x=1}$

oppg. $f(x) = \dfrac{x^3+2x+3}{x^2-1}$ finn asymptoter.

polinomdivisjon

$$\begin{array}{l} (x^3+2x+3) : (x^2-1) = \underline{x + \dfrac{3x+3}{x^2-1}} \\ \underline{x^3 - x} \\ \quad 3x+3 \end{array}$$

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$$\lim_{x\to\infty} f(x) - x = \lim_{x\to\infty} \frac{3(x+1)}{x^2-1} = 0$$

skrå asymptote $\underline{y = x}$

nevner er lik 0 : $x^2-1=0$

$$f(x) = x + \frac{3(x+1)}{(x+1)(x-1)} = x + \frac{3}{x-1}$$

$x = -1$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} x + \frac{3}{x-1} = -1 + \frac{3}{-2} = \underline{-5/2}$$

$x = 1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x + \frac{3}{x-1} = \infty$$

$$\lim_{x\to 1^-} f(x) = -\infty.$$

Vertikal asymptote i $x = 1$.

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Finn asymptotene til $g(x) = \dfrac{x^2+x-2}{x^2-2x+1}$

$$\lim_{x\to\infty} \frac{x^2+x-2}{x^2-2x+1} = \lim_{x\to\infty} \frac{1+1/x-2/x^2}{1-2/x-1/x^2} = 1$$

horisontal asymptote $\underline{y=1}$

nevner lik 0 $\quad x^2-2x+1 = (x-1)^2 = 0$

$x=1.$

$$g(x) = \frac{(x-1)(x+2)}{(x-1)^2} = \frac{x+2}{x-1}$$

så $\displaystyle\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} \frac{x+2}{x-1} = \infty$

$$\lim_{x\to 1^-} \frac{x+2}{x-1} = -\infty$$

Vertikal asymptote i $\underline{x=1}$

Finn asymptote til $h(x) = \begin{cases} \dfrac{1}{x+1} & x < 0 \\ \dfrac{x^2}{x+2} & x \geq 0. \end{cases}$

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$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{1}{x+1} = 0$ $\quad \underline{y = 0 \text{ horisontal asymptote.}}$

$\underline{\text{Vertikal asymptote} \; i \; x = -1}$

polynom divisjon

$$x^2 : x+2 = x - 2 + \frac{4}{x+2}$$
$$\begin{array}{l} \underline{x^2+2x} \\ -2x \\ \underline{-2x-4} \\ \quad 4 \end{array}$$

skrå asymptote $\underline{\underline{y = x-2}}$

$h(x)$ er diskontinuerlig i $x=0$
(hopp disk.)

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$$\frac{x^5}{x^4-3x^3-2}$$ asymptoter?

pol. div.

$$x^5 : x^4-3x^3-2 = x+3+\frac{9x^3+2x+2}{x^4-3x^3-2}$$

$$\begin{array}{l} x^5-3x^4-2x \\ \hline 3x^4+2x \\ 3x^4-9x^3-2 \\ \hline 0 \quad 9x^3+2x+2 \end{array}$$

$y = x+3$ skrå asymptote.

Vertikale asymptoter. i nullpunkt til x^4-3x^3-2
(x^5 er da $\neq 0$)

$x \sim 3.1$
$x \sim -0.8$.

asymptoter til $2x + \frac{1}{x+1} - 1$

vertikal asymptote : $x = -1$.

skrå asymptote $\underline{y = 2x - 1}$.

(18)

asymptoter til $f(x) = \frac{x^3 + x^2}{(x+1)^2}$

$f(x) = \frac{x^2(x+1)}{(x+1)^2} = \frac{x^2}{(x+1)}$.

vertikal asymptote $\underline{x = -1}$.

skrå asymptote : $x^2 : x+1 = x - 1 + \frac{1}{x+1}$

$$\begin{array}{l} x^2 + x \\ \hline -x \\ -x - 1 \\ \hline 1 \end{array}$$

$\underline{y = x - 1}$.

Asymptoter til $g(x) = \dfrac{x^3 - 3x + 2}{(x-1)^2}$

skrå asymptoter

$$
\begin{array}{l}
x^3 - 3x + 2 : x^2 - 2x + 1 = x + 2 \\
\underline{x^3 - 2x^2 + x} \\
\quad 2x^2 - 4x + 2 \\
\quad \underline{2x^2 - 4x + 2} \\
\qquad 0
\end{array}
$$

(19)

$g(x) = x + 2 \qquad x \neq 1.$

skrå asymptote $\underline{y = x + 2}$

ingen vertikale asymptoter.