Produktregelen

28.01.2022

Sammensatt funksjon $(1+x^2)^3$

Produkt av to funksjoner $x^5(1+x)^3$

Produktregelen lar oss finne den deriverte til et produkt $f \cdot g$ fra de deriverte til faktorene

$$\boxed{(f \cdot g)' = f' \cdot g + f \cdot g'}$$

$f(x) = ax+b$ og $g(x) = cx+d$

$f \cdot g(x) = (ax+b)(cx+d) = a \cdot c \cdot x^2 + (ad+bc)x + bd$

$(f \cdot g)' = a \cdot c\,(x^2)' + (ad+bc)\,(x)' + 0$

$= \underline{2x \cdot ac + ad+bc}$

Regner ut direkte:

produktreglen giv: $\left((ax+b)(cx+d)\right)'$

$$\underbrace{(ax+b)'}_{a}(cx+d) + (ax+b)\underbrace{(cx+d)'}_{c}$$

$$= a(cx+d) + (ax+b)\cdot c$$

$$= acx + ad + a\cdot c\cdot x + bc$$

$$= 2x\cdot a\cdot c + ad + bc$$

$$x\cdot\frac{1}{x} = 1$$

$x \neq 0$

$$\left(x\cdot\frac{1}{x}\right)' = (x)'\cdot\frac{1}{x} + x\cdot\left(\frac{1}{x}\right)'$$

$$= 1\cdot\frac{1}{x} + x\cdot\left(\frac{-1}{x^2}\right)$$

$x \neq 0$

$$= \frac{1}{x} - \frac{1}{x} = 0$$

ved produktreglen

Opg

$$(x^r \cdot x^s)' \underset{\text{ved prod. regelen}}{=} (x^r)' \cdot x^s + x^r (x^s)'$$

$$= r x^{r-1} \cdot x^s + x^r s x^{s-1}$$

$$= r x^{r+s-1} + s x^{r+s-1}$$

$$= (r+s) x^{r+s-1}$$

$x^r \cdot x^s = x^{r+s}$ og $(x^{r+s})' = (r+s) x^{r+s-1}$ ✓

Eks $f(x) = \underbrace{x^7}_{1.\text{faktor}} \cdot \underbrace{(x+3)^5}_{2.\text{faktor}}$

produkt regel: $f'(x) = (x^7)'(x+3)^5 + x^7 \cdot ((x+3)^5)'$

$$= 7x^6 (x+3)^5 + x^7 \cdot 5 (x+3)^4 \underbrace{(x+3)'}_{1}$$

$$= x^6 (x+3)^4 [\overbrace{7(x+3)}^{7x+21} \overbrace{+ x \cdot 5}^{+5x}]$$

$$= \underline{x^6 (x+3)^4 (12x + 21)}$$

opg Deriver $g(x) = (x+1)\sqrt{x-1} = (x+1)(x-1)^{1/2}$ $\quad x \geq 1$

$$g'(x) = (x+1)'\sqrt{x-1} + (x+1)\left(\sqrt{x-1}\right)'$$

$$= 1 \cdot \sqrt{x-1} + (x+1)\left(\frac{(x-1)'}{2\sqrt{x-1}}\right)$$

$$= \sqrt{x-1} \cdot \frac{2\sqrt{x-1}}{2\sqrt{x-1}} + \frac{x+1}{2\sqrt{x-1}}$$

$$= \frac{2\left(\sqrt{x-1}\right)^2}{2\sqrt{x-1}} + \frac{x+1}{2\sqrt{x-1}} \qquad x > 1$$

$$= \frac{2(x-1) + x+1}{2\sqrt{x-1}} = \underline{\frac{3x-1}{2\sqrt{x-1}}}$$

$$= (3x-1)/(2\sqrt{x-1})$$

$$\left(\frac{x}{2x^3+1}\right)' = \left(x \cdot \frac{1}{2x^3+1}\right)'$$

$$= (x)' \cdot \frac{1}{2x^3+1} + x \cdot \Big(\overbrace{\frac{1}{2x^3+1}}^{(2x^3+1)^{-1}}\Big)'$$ bruke kjerneregelen

$$= 1 \cdot \frac{1}{2x^3+1} + x \cdot \frac{-1}{(2x^3+1)^2} \cdot (2x^3+1)'$$

$$= \frac{1}{2x^3+1} + \frac{-x \cdot 2 \cdot 3x^2}{(2x^3+1)^2}$$

$$= \frac{2x^3+1 - (6x^3)}{(2x^3+1)^2}$$

$$= \frac{-4x^3+1}{(2x^3+1)^2}$$

$$\left(\frac{x^2}{x^5+4}\right)' = \frac{(x^2)'(x^5+4) - x^2(x^5+4)'}{(x^5+4)^2}$$

$$= \frac{2x(x^5+1) - x^2(5x^4)}{(x^5+4)^2}$$

$$= \frac{2x^6+2x-5x^6}{(x^5+4)^2} = \frac{-3x^6+2x}{(x^5+4)^2}$$

$$\left(\frac{\sqrt{3x+5}}{x^8}\right)' = \frac{\frac{1}{2\sqrt{3x+5}}(3x+5)'\,x^8 - \sqrt{3x+5}\,(x^8)'}{(x^8)^2}$$

$$= \frac{\frac{3}{2\sqrt{3x+5}}\cdot x^8 - \sqrt{3x+5}\cdot 8x^7}{x^{16}}$$

$$= \frac{x^7\left(\frac{3}{2\sqrt{3x+5}} x - 8\sqrt{3x+5}\right)}{x^7\cdot x^9}$$

Kvotientregelen $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

Følger fra produkt og kjerne regelen.

$$\left(\frac{1}{g(x)}\right)' = \left(\left(g(x)\right)^{-1}\right)' \overset{\text{kjerne}}{=} -1\left(g(x)\right)^{-2} \cdot g'(x)$$

kjerne $\quad x^{-1}$ ytre funksjon

$$\underline{\left(\frac{1}{g(x)}\right)' = \frac{-g'(x)}{(g(x))^2}}$$

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' \overset{\text{produkt}}{=} f' \cdot \frac{1}{g} + f\left(\frac{1}{g}\right)'$$

$$= f' \cdot \frac{1}{g} \cdot \frac{g}{g} + f \cdot \left(\frac{-g'}{g^2}\right) = \underline{\frac{f'g - fg'}{g^2}}$$

$$= \frac{1}{x^9}\left(\frac{3\cdot x - 8\cdot 2(3x+5)}{2\sqrt{3x+5}} \right)$$

$$= \frac{3x - 48x - 80}{2x^9\sqrt{3x+5}} = \frac{-45x-80}{2x^9\sqrt{3x+5}}$$

($\left(\sqrt{3x+5}\cdot x^{-8}\right)'$ produktregel ... minst like enkelt)

opg Deriver $\frac{x^2}{x^5-7}$

$$\left(\frac{x^2}{x^5-7}\right)' = \frac{(x^2)'(x^5-7) - x^2(x^5-7)'}{(x^5-7)^2}$$

$$= \frac{2x(x^5-7) - x^2(5x^4)}{(x^5-7)^2}$$

$$= \frac{2x^6 - 14x - (5x^6)}{(x^5-7)^2}$$

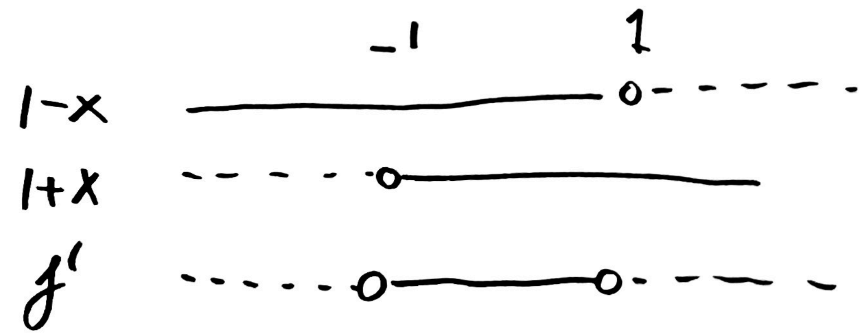
$$= \frac{-3x^6 - 14x}{(x^5-7)^2}$$

Funksjonsdrøfting av $f(x) = \frac{x}{x^2+1}$

odde funksjon
$f(-x) = -f(x)$
symmetrisk om origo.

$$f'(x) = \frac{(x)'(x^2+1) - x(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}$$



f' vokser $[-1, 1]$
avtar $(-\infty, -1]$ og $[1, \infty)$.

toppunkt : $\left(1, \frac{1}{2}\right)$

bunnpunkt : $\left(-1, -\frac{1}{2}\right)$

$$f''(x) = \left(\frac{1-x^2}{(1+x^2)^2}\right)' = \left((1-x^2)(1+x^2)^{-2}\right)'$$

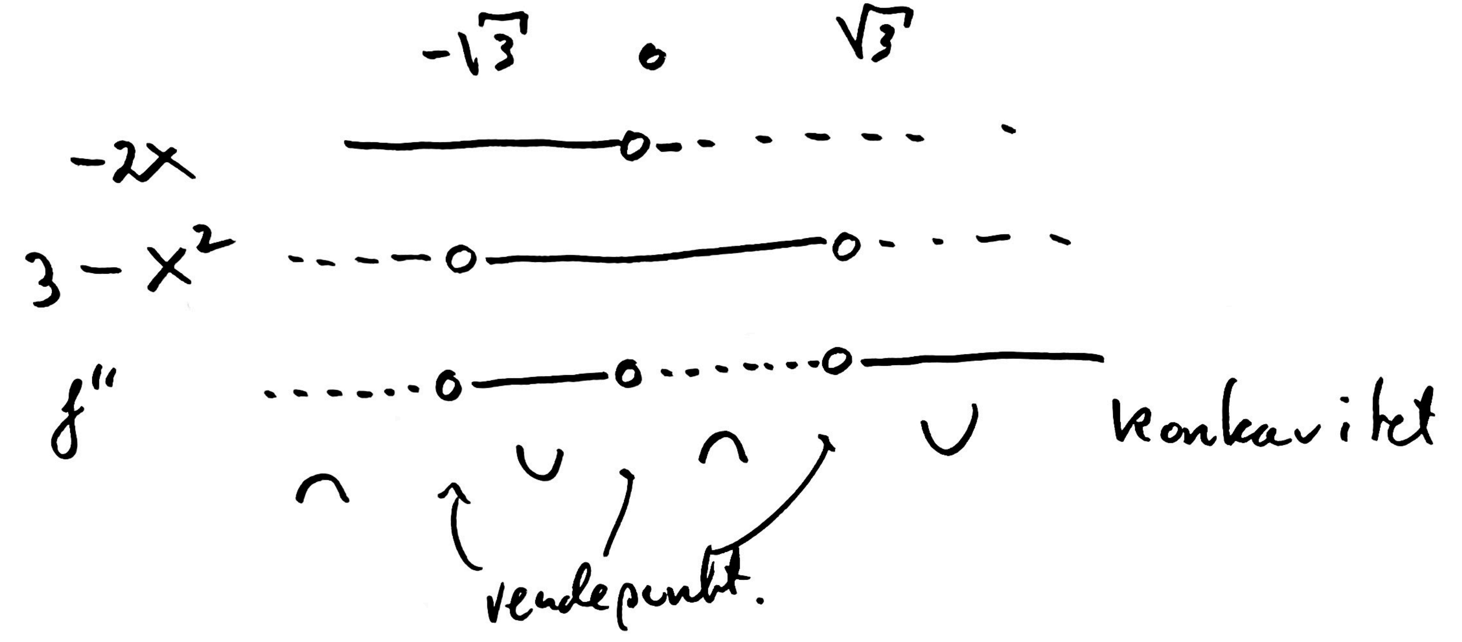
$$= (1-x^2)'(1+x^2)^{-2} + (1-x^2)\left(\left((1+x^2)^{-2}\right)'\right)$$

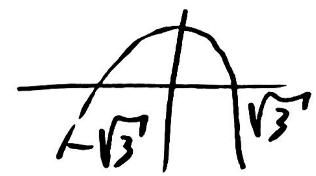
$$= -2x(1+x^2)^{-2} + (1-x^2)\left(-2(1+x^2)^{-3}\underbrace{(1+x^2)'}_{2x}\right)$$

$$= \frac{-2x}{(1+x^2)^2}\frac{1+x^2}{1+x^2} - \frac{2(1-x^2)\cdot 2x}{(1+x^2)^3}$$

$$= \frac{-2x\left(1+x^2+2(1-x^2)\right)}{(1+x^2)^3} = \frac{-2x(3-x^2)}{(1+x^2)^3}$$

$f''(x) = 0$ for $x = -\sqrt{3}$, $x = 0$ og $x = \sqrt{3}$





$$\Delta f = f(x+h) - f(x)$$

Bevisskisse for produktregelen

$$(f \cdot g)'(x) = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x) \cdot g(x)}{h}$$

$$= \lim_{\Delta x \to 0} \frac{(f(x) + \Delta f)(g(x) + \Delta g) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x)\,\Delta g + g(x) \cdot \Delta f + \Delta g \Delta f}{\Delta x}$$

$$= \underline{f(x) \cdot g'(x) + f'(x) \cdot g(x)} \ (+0)$$

$h = \Delta x$

$f(x + \Delta x) = f(x) + \Delta f$

28.01
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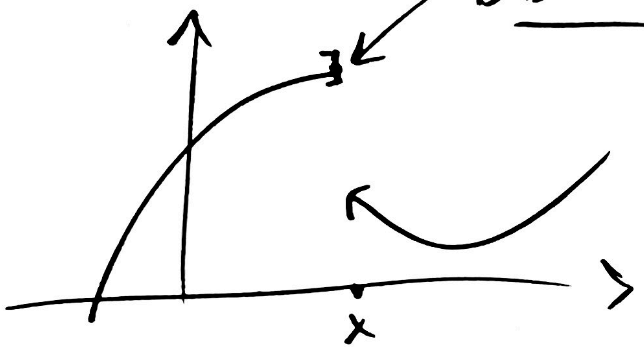
Øving.



Diskontinuitet. : Et punkt hvor grafen (funksjonen) ikkje er kontinuerlig.

Naturlig definisjonsmengde til et uttrykk i x
alle verdier x slik at uttrykket gir mening.

$f = \frac{1}{x}$ $\quad D_f = \mathbb{R} \setminus \{0\} = \langle -\infty, 0 \rangle \cup \langle 0, \infty \rangle$

Alle x ulik 0.

$g = \sqrt{x}$ $\quad D_g = [0, \infty \rangle$.

$h = \frac{1}{\sqrt{x-3}}$ $\quad D_h = \langle 3, \infty \rangle$

opg 7.219 (cosin) $f(x) = 2x^3 + ax^2 - 4x - 2$

familie av polynomer.

Bestem a slik at f har bunnpunkt for $x = 2$.

Kritiske punkter er de stasjonære punktene. (ingen endepunkt, $f(x)$ deriverbar for alle x)

$$f'(x) = 2 \cdot 3x^2 + a \cdot 2x - 4$$
$$= 2(3x^2 + ax - 2).$$

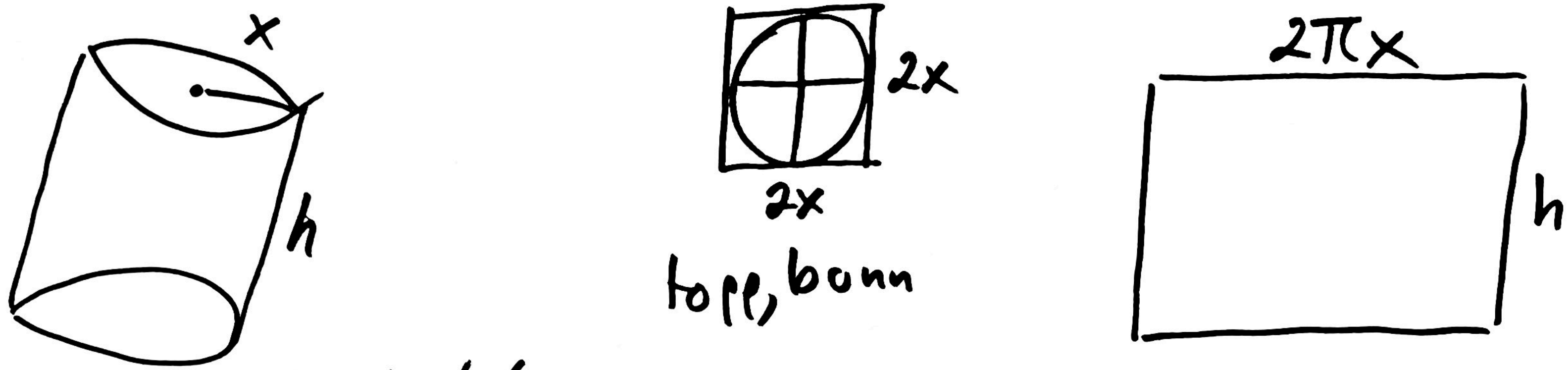
Bunnpunkt: $x = 2 \Rightarrow f'(2) = 0$:

$$2(3 \cdot 2^2 + 2 \cdot a - 2) = 0$$
$$12 + 2a - 2$$
$$10 + 2a = 0$$
$$\underline{a = -5}.$$

$f''(x) = 2(6x + a)$ $f''(2) = 2(12 + a)$

For $a = -5$ er $f''(2) = 14 > 0$ så f har bunnpunkt hvor $x = 2$ for $\underline{a = -5}$.

opg 7.202



Volumet $= \pi x^2 \cdot h$ (enhet dm^3 = Liter)

x, h enhet desimeter

$$V = \pi x^2 h = 1 \text{ (Liter)} \quad \text{gir } h = \frac{1}{\pi x^2}$$

Areal til materialet som benyttes til å lage boksen

$$A = \underline{2 \cdot (2x)^2 + 2\pi x \cdot h}$$
$$= 8x^2 + \frac{2}{x}$$

Finn Bunnpunktet til $A(x)$ ved regning.

$$A(x) = 8x^2 + \frac{2}{x} = 2\left(4x^2 + \frac{1}{x}\right)$$

$$A'(x) = 2\left(4 \cdot 2x + \frac{-1}{x^2}\right) = 2\,\frac{8x^3 - 1}{x^2}$$

$$A'(x) = 0$$

$$\Leftrightarrow \quad 8x^3 = 1$$

$$x^3 = \frac{1}{8} = \left(\frac{1}{2}\right)^3$$

Så $\underline{x = \frac{1}{2}}$

Arealet er da lik $A(\frac{1}{2}) = 8 \cdot \left(\frac{1}{2}\right)^2 + \frac{2}{1/2}$

$$= 8 \cdot \frac{1}{4} + 2 \cdot 2 = 2 + 4 \underline{= 6}$$

7.210 $f(x) = \frac{1}{3}x^3 + x^2 + 2x \qquad D_f = \mathbb{R}$

Når stigningstallet til tangentlinjene minst?

$f'(x) = x^2 + 2x + 2 = (x+1)^2 + 1$

$f'(x)$ minst for $\underline{x = -1}$. verdien er da lik $f'(-1) = 1$.

$(f'(x))' = f''(x) = 0$

$2x + 2 = 0$

$x = -1.$

Stigningstallet til tangentlinjen er minst i punktet

$(-1, f(-1)) = (-1, \frac{-1}{3} + 1 - 2)$

$= \underline{(-1, \frac{-4}{3})}$

Eksempel på derivasjon

$$\left(x\sqrt{2-x}\right)' = (x)'\sqrt{2-x} + x\left(\sqrt{2-x}\right)'$$

$$= \sqrt{2-x} + x\left(\frac{1}{2\sqrt{2-x}}\underbrace{(2-x)'}_{-1}\right)$$

$$= \sqrt{2-x} - \frac{x}{2\sqrt{2-x}} \quad \ldots$$

$$\left(x^3+2x^2+5\right)' = 3x^2 + 4x + 0$$

$$\left((1+3x)^5\right)' = 5(1+3x)^4\underbrace{(1+3x)'}_{3}$$

lett å glemme
å gange med
den deriverte til
kjernen.

$$\left(\frac{1}{(\sqrt{x}+4)^2}\right)' = \left((x^{1/2}+4)^{-2}\right)'$$

$$= -2\,(x^{1/2}+4)^{-3}\cdot(x^{1/2}+4)'$$

$$= -2\,\frac{1}{(\sqrt{x}+4)^3}\left(\frac{1}{2}\cdot x^{-1/2}\right)$$

$$= \underline{\frac{-1}{\sqrt{x}\,(\sqrt{x}+4)^3}}\,.$$

$$\frac{1}{x^2+1}$$

ikkje naturlig å benytte kvotientregel

$$\frac{\overset{0}{\widetilde{1'}}\cdot(x^2+1) - (x^2+1)'}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2}\,.$$

Mer naturlig: $\underset{\text{kjerneregel}}{\left((x^2+1)^{-1}\right)'} = -1\,(x^2+1)'\,(x^2+1)^{-2} = \underline{\frac{-2x}{(x^2+1)^2}}\,.$