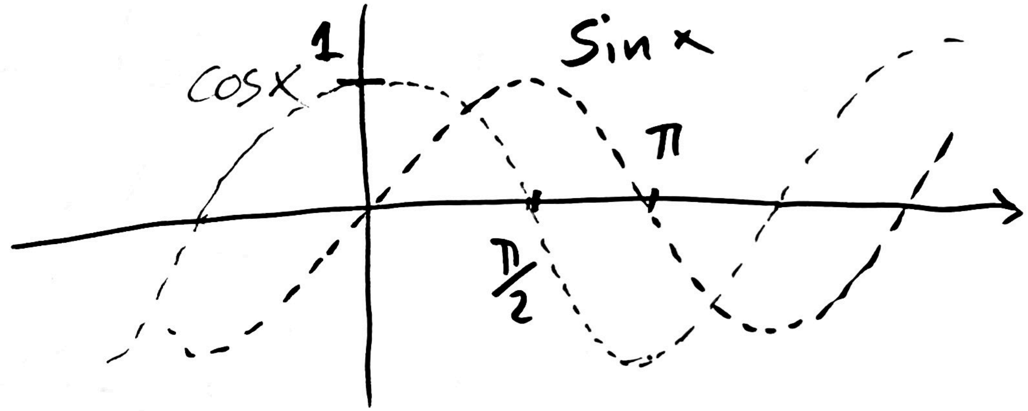
18.02 2022 11.8-9 Deriverte av trigonometriske funksjoner

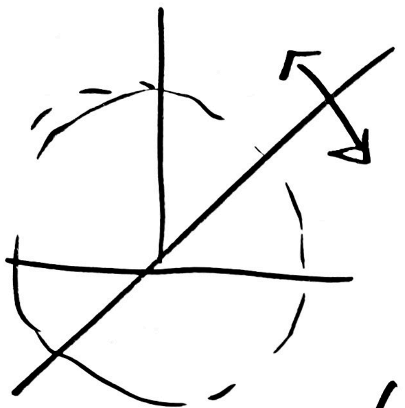
$$\underline{(\sin x)' = \cos x}$$ x radianer.



Eksempel

$$(3\sin(5x-1))'$$
$$= 3\cos(5x-1)\cdot(5x-1)'$$
$$= \underline{15\cos(5x-1)}$$

$$(x\sin x)' \underset{\text{regel}}{\overset{\text{prod.}}{=}} (x)'\sin x + x(\sin x)'$$
$$= \underline{\sin(x) + x\cos x}$$



$$\cos\left(\frac{\pi}{2}-v\right) = \sin v$$

$$\sin\left(\frac{\pi}{2}-v\right) = \cos v$$

$$(\cos x)' = \left(\sin\left(\frac{\pi}{2}-x\right)\right)'$$

$$= \cos\left(\frac{\pi}{2}-x\right) \cdot \underbrace{\left(\frac{\pi}{2}-x\right)'}_{-1}$$

$$(\cos x)' = -\cos\left(\frac{\pi}{2}-x\right) = -\sin x$$

$$\boxed{(\cos x)' = -\sin x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad (\cos x \neq 0)$$

$$(\tan x)' = \left(\sin x \cdot (\cos x)^{-1}\right)'$$

$$= (\sin x)' \cdot \frac{1}{\cos x} + \sin x \left((\cos x)^{-1}\right)'$$

$$= \cos x \cdot \frac{1}{\cos x} + \sin x \left(-(\cos x)^{-2} \cdot \underbrace{(\cos x)'}_{-\sin x}\right)$$

$$= \frac{\cos x}{\cos x} + (-1)^2 \frac{\sin^2 x}{\cos^2 x}$$

$$= 1 + \left(\frac{\sin x}{\cos x}\right)^2 = \underline{1 + \tan^2 x}$$

"felles nevner"

$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \underline{\frac{1}{\cos^2 x}}$$

$$(\tan x)' = 1 + \tan^2 x = \frac{1}{\cos^2 x}.$$

opg Deriver $f(x) = 2\sin x \cdot \cos x$

Benytter prod.regelen: $f'(x) = 2\Big(\underbrace{(\sin x)'}_{\cos x}\cos x + \sin x \cdot \underbrace{(\cos x)'}_{-\sin x}\Big)$

$$= \underline{2(\cos^2 x - \sin^2 x)}$$

Alternativt: $f(x) = \sin(2x)$

$f'(x) = (\sin(2x))' \overset{\text{kjerne-regel}}{=} \cos(2x) \cdot \underbrace{(2x)'}_{2}$

$= 2\cos(2x)$

Svarene er like siden $\cos(2x) = \cos^2 x - \sin^2 x$ for alle x.

Vi har $g(x) = \sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$$g'(x) = (\sin^2 x)' = ((\sin x)^2)' = 2(\sin x) \cdot \underbrace{(\sin x)'}_{\cos x}$$

$$= 2\sin x \cdot \cos x = \underline{\sin(2x)}$$

Alternativt: $g'(x) = \left(\frac{1}{2}(1 - \cos(2x))\right)'$

$$= \frac{1}{2}\Big(\underbrace{(1)'}_{0} - \underbrace{(\cos(2x))'}_{-\sin(2x)\cdot(2x)'}\Big)$$

$$= \frac{1}{2} \cdot 2\left((-1)^2 \sin(2x)\right)$$

$$= \underline{\sin(2x)}$$

$$f(x) = e^{-x}\cos x = \frac{\cos x}{e^{x}}$$

$$f'(x) = (e^{-x})'\cos x + e^{-x}(\cos x)'$$

$$= e^{-x}\cdot(-x)'\cos x + e^{-x}(-\sin x)$$

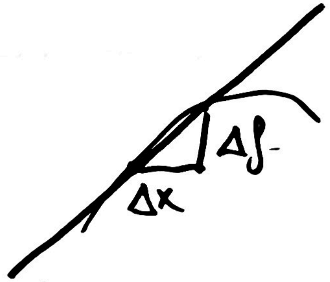
$$= e^{-x}(-\cos x - \sin x)$$

$$= -e^{-x}(\underbrace{\cos x + \sin x}_{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)})$$

$$= \underline{-\sqrt{2}\,e^{-x}\sin\left(x+\frac{\pi}{4}\right)}$$

Bevis for $(\sin x)' = \cos x$

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$



Addisjonsformel for sin: $\sin(x+h) = \cos x \cdot \sin(h) + \cos(h) \cdot \sin x$

setter inn:

$$(\sin x)' = \lim_{h \to 0} \frac{1}{h}\left[\cos x \cdot \sin(h) + \sin x(\cos(h) - 1)\right]$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\sin(h)}{h} + \sin x \lim_{h \to 0} \frac{\cos(h) - 1}{h}$$

$$\boxed{\text{Resultat: } \lim_{h \to 0} \frac{\sin(h)}{h} = 1}$$

Denne grensen fører til $\underline{\lim_{h \to 0} \frac{1-\cos(h)}{h^2} = \frac{1}{2}}$

$$\lim_{h \to 0} \frac{1-\cos(h)}{h^2} = \lim_{h \to 0} \frac{(1-\cos(h))(1+\cos(h))}{h^2\,(1+\cos(h))}$$

$$= \lim_{h \to 0} \frac{1-\cos^2(h)}{h^2} \cdot \frac{1}{1+\cos(h)} = \lim_{h \to 0} \left(\frac{\sin(h)}{h}\right)^2 \cdot \frac{1}{1+\cos(h)}$$

$$= \left(\lim_{h \to 0} \frac{\sin(h)}{h}\right)^2 \cdot \lim_{h \to 0} \frac{1}{1+\cos(h)} = 1^2 \cdot \frac{1}{1+1} = \underline{\underline{\frac{1}{2}}} \checkmark$$

Dette impliserer: $\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \frac{-(1-\cos(h))}{h^2} \cdot h$

$$= \left(-\lim_{h \to 0} \frac{1-\cos(h)}{h^2} \cdot \lim_{h \to 0} h\right) = -\frac{1}{2} \cdot 0 = 0$$

Dette gir $\left(\sin(x)\right)' = \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} + \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{0}$

$$\underline{\left(\sin(x)\right)' = \cos(x)}$$

Vi viser nå at $\lim_{h\to 0} \frac{\sin h}{h} = 1$

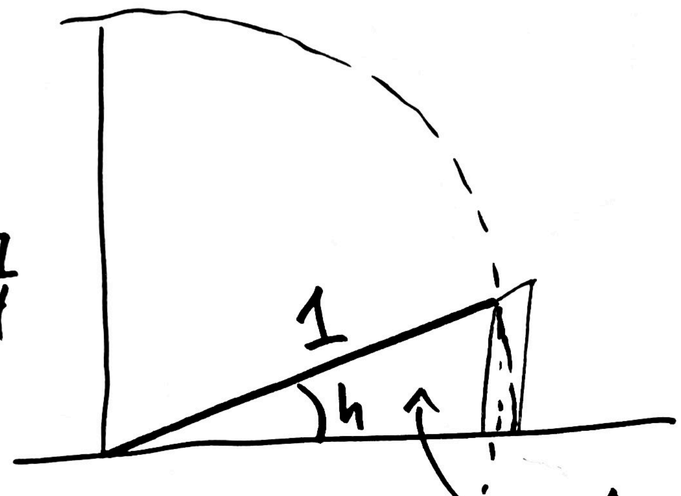
$$\frac{\sin(-h)}{(-h)} = \frac{-\sin(h)}{-h} = \frac{\sin h}{h}$$

Derfor er $\lim_{h\to 0^-} \frac{\sin(h)}{h} = \lim_{h\to 0^+} \frac{\sin(h)}{h}$.

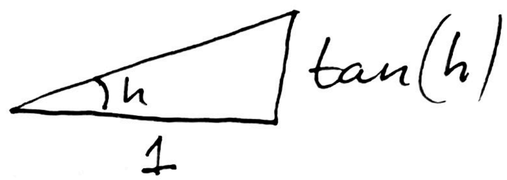
Det er derfor tilstrekkelig å vise

$$\lim_{h\to 0^+} \frac{\sin h}{h} = 1.$$

$0 < h < \frac{\pi}{4}$



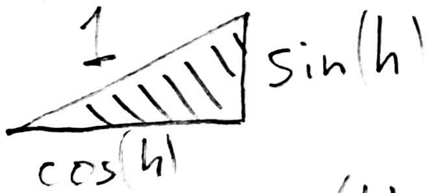
Areal $\frac{1}{2} \cdot 1 \cdot \tan(h)$



Areal sirkelsegment

$\frac{1}{2}$ buelengde

$= \frac{1}{2} h$ siden vinkelen er i radianer.



Areal $\frac{1}{2} \sin(h) \cdot \cos(h)$

$h > 0$

$$\frac{1}{2} \sin(h) \cdot \cos(h) < \frac{1}{2} h < \frac{1}{2} \tan(h)$$

Ganger med 2 og deler med h (positiv!)

$$\frac{\sin(h)}{h} \cdot \cos(h) < 1 \quad \text{og} \quad 1 < \frac{\sin(h)}{h} \cdot \frac{1}{\cos(h)}$$

$\cos(h) > 0$ så

$$\cos(h) < \frac{\sin h}{h} < \frac{1}{\cos(h)}$$

Siden $\lim_{h \to 0^+} \cos(h) = 1$

og $\lim_{h \to 0^+} \frac{1}{\cos(h)} = 1$

så må $\lim_{h \to 0^+} \frac{\sin(h)}{h} = 1$

Deriverte til $\sin x$ hvis vinkelen er i grader:

x i grader x°

vinkel i radian er $\frac{\pi x^\circ}{180^\circ}$

$$\left(\sin\left(\frac{\pi \cdot x^\circ}{180^\circ}\right)\right)' = \cos\left(\frac{\pi \cdot x^\circ}{180^\circ}\right) \cdot \left(\frac{\pi x}{180^\circ}\right)'$$

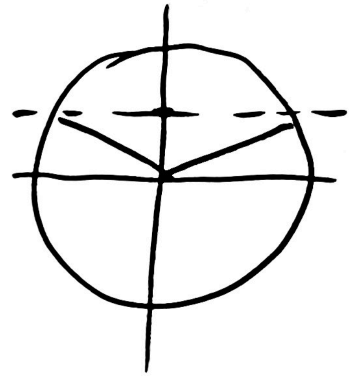
$$= \underline{\frac{\pi}{180^\circ} \cos\left(\frac{\pi x^\circ}{180^\circ}\right)}$$

Vi unngår derfor bruk av enhet grader når trigonometriske funksjoner skal deriveres!

$g(x) = 2\cos x + x$

$g'(x) = 2(\cos x)' + (x)' = \underline{-2\sin x + 1}$

Monotoniegenskaper og ekstremalpunkt.



$g'(x) = 0$ når $2\sin(x) = 1$

$\sin(x) = 1/2$

$x = \frac{\pi}{6} + 2\pi \cdot n$ og $x = \frac{5\pi}{6} + 2\pi n$

$g'(x) = 2\left(\frac{1}{2} - \sin(x)\right) > 0$ når $\sin(x) < \frac{1}{2}$

$g(x)$ øker i $\left[\frac{5\pi}{6}, \frac{13\pi}{6}\right]$ opp til hele omløp

avtar i $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ opp til hele omløp

$g(x)$ avtar fra $\frac{\pi}{6}$ og stiger frem til $\frac{\pi}{6}$. $\Rightarrow$

Toppunkt: $\left(\frac{\pi}{6} + 2\pi \cdot n \, , \, g\left(\frac{\pi}{6} + 2\pi \cdot n\right)\right)$

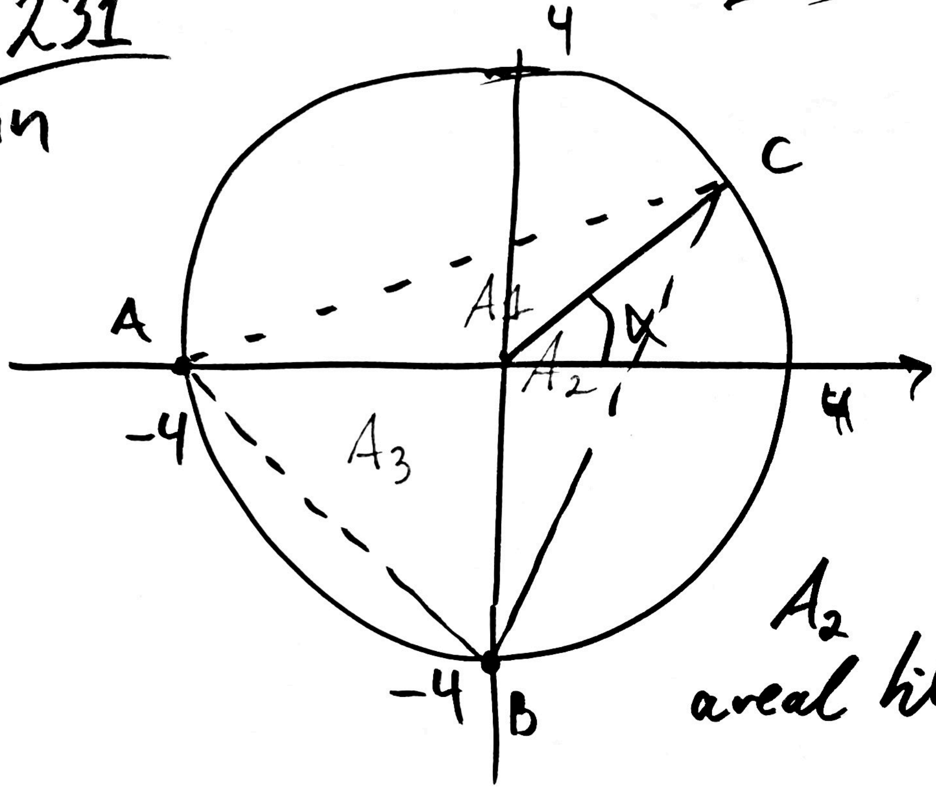
$\left(\frac{\pi}{6} + 2\pi \cdot n \, , \, \frac{\sqrt{3}}{2} + \frac{\pi}{6} + 2\pi \cdot n\right)$

Tilsvarende bunnpunkt: $\left(\frac{5\pi}{6} + 2\pi \cdot n \, , \, g\left(\frac{5\pi}{6} + 2\pi \cdot n\right)\right)$

øving

11.231

cosin



Areal ΔABC som en funksjon av α

$A_3 = \frac{1}{2}(4 \cdot 4) = \underline{8}$

A_2 areal til

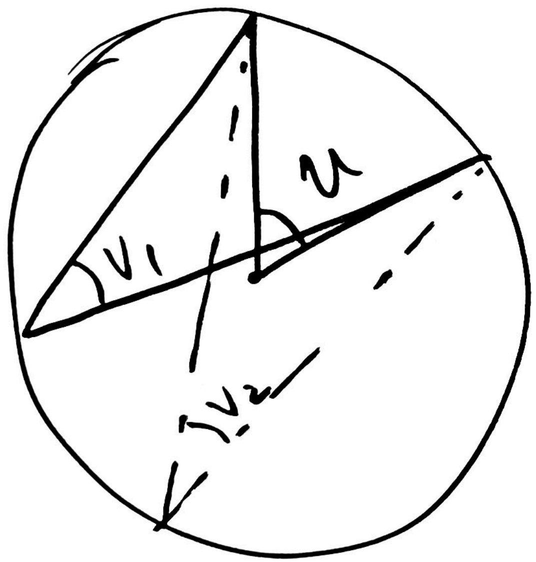
$$A_2 = \frac{1}{2} 4 \cdot 4 \cdot \underbrace{\sin\left(\frac{\pi}{2} + \alpha\right)}_{\underbrace{\sin\frac{\pi}{2}}_{1} \cdot \cos(\alpha) + \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\sin\alpha}$$

$$A_2 = \underline{8\cos\alpha}.$$

Tilsvarende $A_1 = 8\cos\left(\frac{\pi}{2} - \alpha\right) = \underline{8\sin(\alpha)}$

Areal ΔABC er

$$A = 8(1 + \cos\alpha + \sin\alpha).$$



u sentralvinkel

v_1, v_2 etc periferivinkler

Resultat

$$\underline{u = 2v}$$

sentralvinkel $= 2 \cdot$ periferivinkel

spesielt