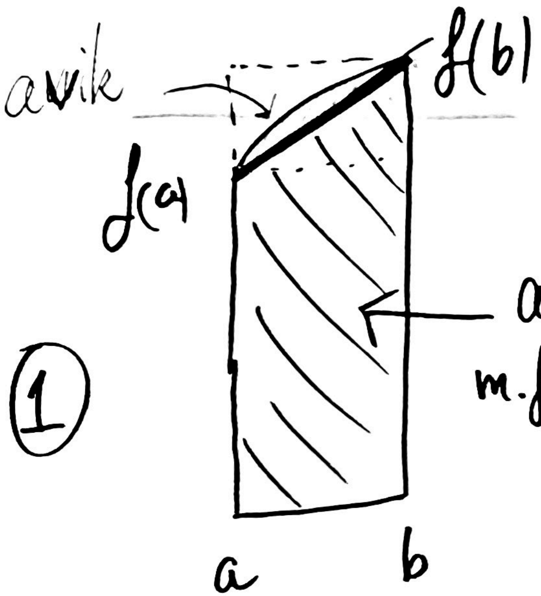
11.03
2022

Numerisk integrasjon

Fausk

Trapesmetoden

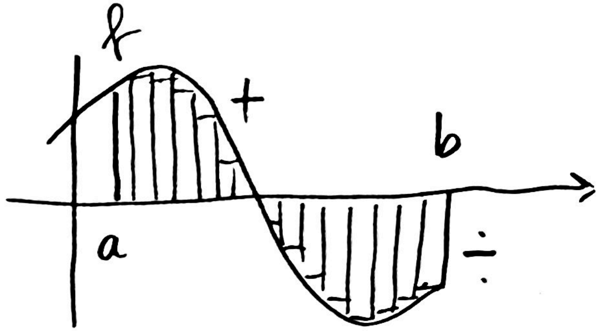


areal m. fortegn $(b-a)\left(\frac{f(b)+f(a)}{2}\right)$

(1)

Trapesmetoden gir eksakt verdi når $f(x)$ er en lineærfunksjon

$f(x) = ax + b$ (polynom av grad 1)

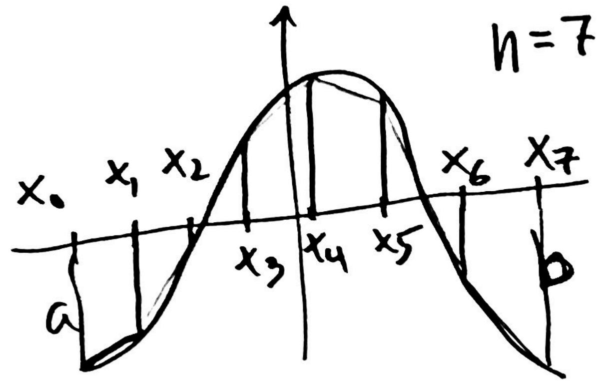


$$\int_a^b f\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot f(x_i)$$

$$a+(i-1)\frac{b-a}{n} \le x_i \le a + i\left(\frac{b-a}{n}\right)$$

f kontinuerlig i $[a, b]$

(2)



$n=7$

$x_0 = a$

$x_1 = a+d$

$x_2 = a+2d$

$\vdots$

$$\underline{x_i = a + i \cdot d}$$

så

$$x_n = a + n\,\frac{b-a}{n} = \underline{b}$$

$d = \dfrac{b-a}{n}$ bredden til hvert av trapesene.

T_n tilnærming til $\int_a^b f(x)\,dx$ med n like intervaller.

$$T_n = d\left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right)$$

$$= d\left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right)$$

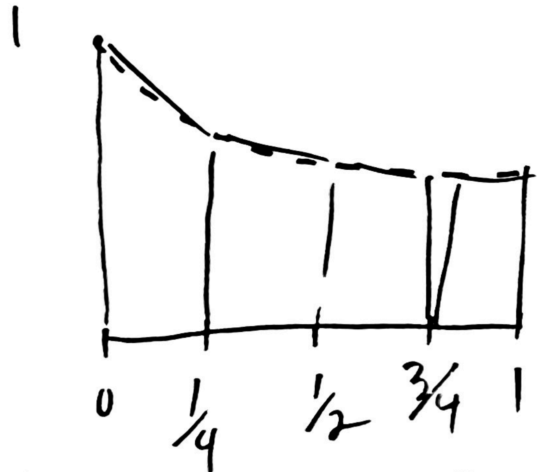
Resultat $\left|\int_a^b f(x)\,dx - T_n\right| \le \dfrac{M_2\,(b-a)^3}{n^2} = M_2\left(\dfrac{b-a}{n}\right)^2 (b-a)$

$$M_2 = \max_{x\in[a,b]} |f''(x)|$$

~~opg~~ eks. Estimer $\int_0^1 \frac{1}{1+x^2}\,dx$ $\left(\text{eksakt verdi } \frac{\pi}{4}\right)$

med trapesmetoden og 4 intervaller.

(3)



$d = \frac{b-a}{4} = \frac{1}{4}$

$$T_4 = \frac{1}{4}\left(\frac{1}{2}f(0) + f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4}) + \frac{1}{2}f(1)\right)$$

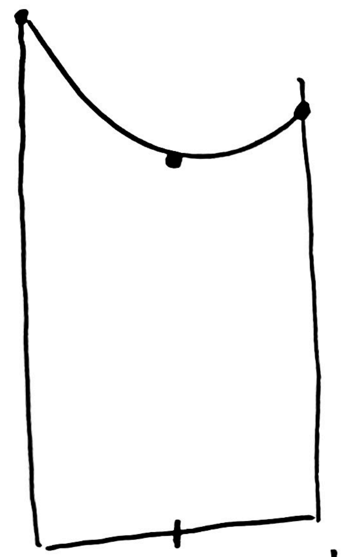
$$= \frac{1}{4}\left(\frac{1}{2}\cdot 1 + \frac{1}{1+(1/4)^2} + \frac{1}{1+(1/2)^2} + \frac{1}{1+(3/4)^2} + \frac{1}{4}\right)$$

$$= \frac{1}{4}\left(\frac{3}{4} + \frac{4^2}{4^2+1} + \frac{4^2}{4^2+4} + \frac{4^2}{4^2+9}\right)$$

$$\sim 0.78279\ldots$$ (tilnærming til en brøk)

(4)

Simpsons metode



$a \quad \frac{b+a}{2} \quad b$

Vekting $1 \quad 4 \quad 1$

$$S_2 = \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{6} \cdot (b-a)$$

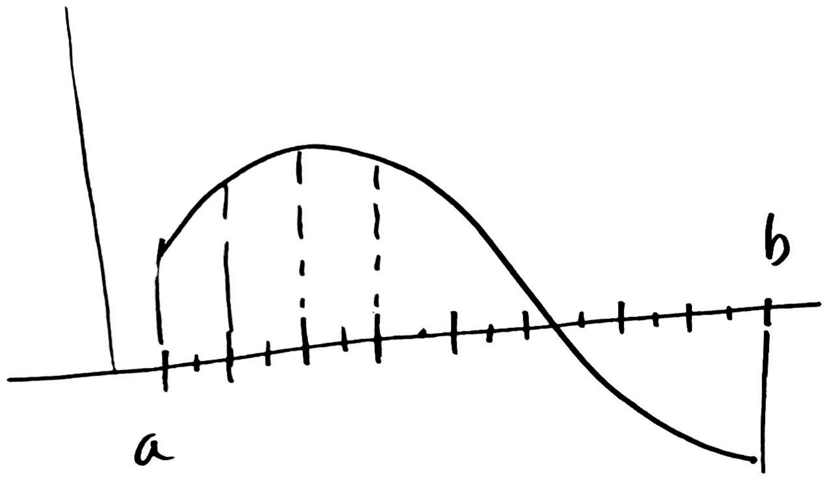
$$= \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{3} \cdot \left(\frac{b-a}{2}\right)$$

$$d = \frac{b-a}{2}$$

S_2 gir eksakt integral for 2.grads uttrykk.

Tilstrekkelig å sjekke for $f(x) = x^2$

(5)



m dobbelintervaller

$n = 2m$ intervaller

$$S_n = \frac{1}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{2n-1}) + f(x_{2n}) \Big) \left(\frac{b-a}{n} \right)$$

1 4 1
  1 4 1
    1 4 1 ... etc

vekting 1 4 2 4 2 4 2 4 ... 4 1

Result: $\left| \int_a^b f(x)\,dx - S_n \right| \le \frac{M_4}{180} \frac{(b-a)^5}{n^4}$

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \underline{\tfrac{1}{3}(b^3 - a^3)}$$

$$S_2 = \frac{b-a}{6}\Big(a^2 + \underbrace{4\cdot\Big(\frac{b+a}{2}\Big)^2}_{(b+a)^2} + b^2\Big) = \frac{b-a}{6}\big(a^2 + a^2 + 2ab + b^2 + b^2\big)$$

6

$$= \frac{b-a}{6}\,2\,(a^2+ab+b^2) = \tfrac{1}{3}(b-a)(a^2+ab+b^2)$$

$$= \tfrac{1}{3}\big(\,ba^2 + ab^2 + b^3 - a^3 - a^2b - ab^2\big)$$

$$= \tfrac{1}{3}(b^3 - a^3)$$

så $\int_a^b x^2\,dx = S_2$

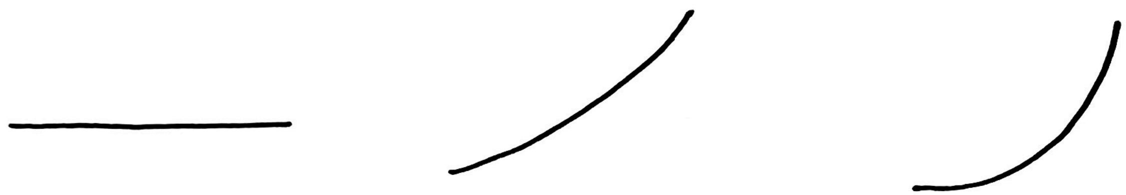
sjekk gjerne at $\int_a^b x^3\,dx = S_2$

Simpsons metode gir eksakt verdi for alle 3. grads polynomer !

ØVING

- f økende i $[a, b]$ hvis $f(x) \leq f(y)$ når $x \leq y$

⑦



- Ekte økende i $[a, b]$: $f(x) < f(y)$ når $x < y$

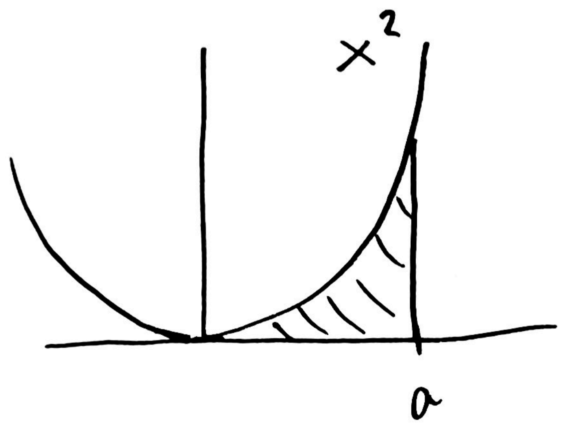
x^2 økende i $[0, 1]$
ekte økend —‖—

$f(x) = 2$ alle x
$f(x)$ økende
men ikkje ekte økende.

15.228

(8)

a)



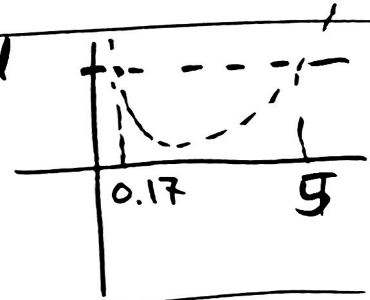
$$\int_0^a x^2\,dx = 9 \qquad a>0$$

$$\int_a^0 x^2\,dx = 9 \qquad a<0$$

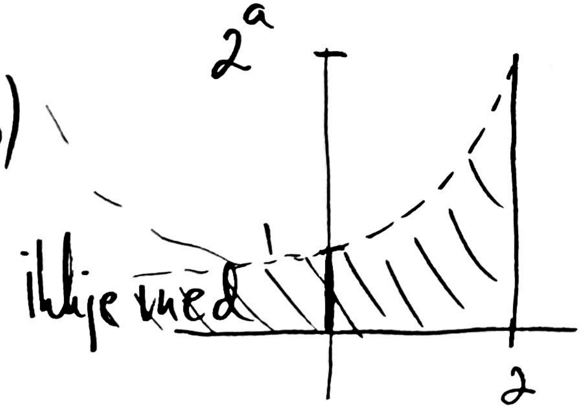
$$\left|\frac{a^3}{3}\right| = 9$$

$$a = \underline{\pm 3}$$

grafen til $\frac{2^x}{x}$



b)



$$A = \frac{32}{5} = \int_0^2 x^a\,dx \qquad a \neq -1$$

(>6)

$$\left.\frac{x^{a+1}}{a+1}\right|_0^2 \qquad a>0$$

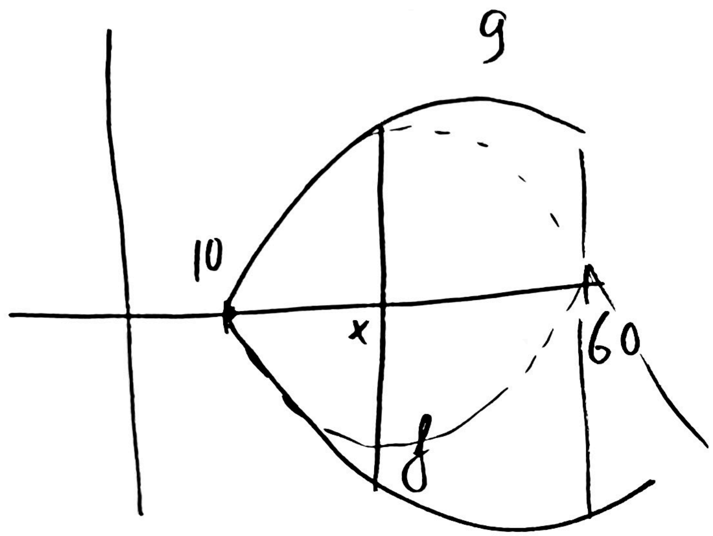
$$\frac{32}{5} = \frac{2^{a+1}}{a+1}$$

$$\underline{a=4}$$

$$a \approx \underline{-0.823404\ldots}$$

Løsningen $a \approx -0.82$ er glemt i LF iboken.

9



$f(10) = 0$ $\qquad f'(10) = -3$

$g(10) = 0$ $\qquad g'(10) = 7$

b) $g > f$

$d(x) = g(x) - f(x)$

c) $d'(x) = 0$ når $d(x)$ er størst mulig.

$d' = \frac{1}{10}\left[-(2x-50) - (2x-90)\right]$

$= \frac{-1}{10}\left[4x - 50 - 90\right]$ gir $x = \frac{140}{4} = \frac{70}{2}$

$= \underline{\underline{35}}$

Vi benyttet geogebra til å estimere integraler
med trapes-metoden i timene.
(10) sjekk kommandoene: Trapezoidal(...) og Integral(...)

Til simpsons metode benyttet vi et
lite program i språket Python.
Dette kan også gjøres i geogebra, men da på en
mer klossete måte. (Rekker etc)

Hvis dere har overskudd og interesse kan dere
undersøke Python mer selv. Dette er ikkje pensum.

Anaconda.org (en god måte å laste ned Python + mer på)
Python.org
W3schools.com (opplæring. Tillater å kjøre kode inntilleser.)