16.03
2022

16.2-3 Variabelskifte
substitusjon

Kjerne regelen

$$\left(F(u(x))\right)' = F'(u(x)) \cdot u'(x)$$

(1)

$$\frac{d\,F(u(x))}{dx} = \frac{d\,F(u)}{d\,u} \cdot \frac{du}{dx}$$

$$\left(e^{-x^2}\right)' = e^{-x^2} \cdot (-x^2)'$$

$$= -2x\,e^{-x^2}$$

Anta $F(x)$ er en antiderivert til $f(x)$. $F'(x) = f(x)$

$$\left(F(u(x))\right)' = \underbrace{f(u(x))}_{F'(u(x))} \cdot u'(x)$$

Så $\int f(u(x)) \cdot u'(x)\,dx = F(u(x)) + C$

$$= \int f(u)\,du \quad (= F(u(x)) + C)$$

$u'(x)\,dx$ erstattes av du

(2) $\frac{du}{dx}\,dx$ . . . du

Eks $\int \underbrace{-2x}_{u'(x)} \underbrace{e^{-x^2}}_{e^u} dx$ hvor $u = -x^2$

$u' = -2x$

$= \int e^u \, du$

$= e^u + c$

$= \underline{e^{-x^2} + c}$

(alternativt $v = x^2 \ldots$)

$u = 1 + x^2$

$\int \frac{x}{1+x^2} dx$

$u' = 2x$

så $x = \frac{u'}{2}$

$= \int \frac{1}{u} \cdot \frac{1}{2} \cdot \underbrace{u'\,dx}_{du}$

$= \frac{1}{2} \int \frac{1}{u} du \quad = \frac{1}{2} \ln|u| + c$

$= \underline{\frac{1}{2} \ln|1+x^2| + c}$

opg
$$\int x^2 \underbrace{(4+5x^3)^7}_{u^7} \, dx$$

$$= \int \frac{1}{15} u' \quad u^7 \, dx \qquad \left(\begin{array}{l} u = 4+5x^3 \\ u' = 15x^2 \\ x^2 = u' \cdot \frac{1}{15} \end{array} \right)$$

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$$= \frac{1}{15} \int u' \, u^7 \, dx$$

$$= \frac{1}{15} \int u^7 \, du$$

$$= \frac{1}{15} \cdot \frac{u^8}{8} + C$$

$$= \frac{1}{120} (4+5x^3)^8 + C$$

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ex
$$\int x^5 (1+x^3)^7 \, dx \qquad \begin{array}{l} u = 1+x^3 \\ u' = 3x^2 \\ x^3 = u-1 \end{array}$$

$$= \int \underbrace{3x^2}_{u'} \cdot \frac{1}{3} x^3 (1+x^3)^7 \, dx$$

$$u' \cdot \frac{1}{3}(u-1) \cdot u^7 \, dx$$

$$= \frac{1}{3} \int (u-1) \cdot u^7 \, du = \frac{1}{3} \int u^8 - u^7 \, du$$

$$= \frac{1}{3}\left(\frac{u^9}{9} - \frac{u^8}{8} \right) + C = \frac{1}{3}\left(\frac{(1+x^3)^9}{9} - \frac{(1+x^3)^8}{8} \right) + C$$

$$= \frac{1}{3 \cdot 9 \cdot 8} (1+x^3)^8 \left(8(1+x^3) - 9 \right)$$

$$= \frac{(8x^3 - 1)}{27 \cdot 8} (1+x^3)^8$$

(4)

$u = \sin x$

$u' = \cos x$

$$\int \underbrace{\cos(x)}_{u'} \cdot \underbrace{\sin^5(x)}_{u^5} \, dx$$

$$= \int u^5 \, du = \frac{u^6}{6} + C = \frac{1}{6} \sin^6 x + C$$

$$\int \cos^3(x) \cdot \sin^5(x) \, dx$$

$$= \int \underbrace{\cos x}_{u'} \cdot \underbrace{\cos^2 x}_{1-u^2} \cdot \underbrace{\sin^5 x}_{u^5} \, dx$$

(Pytagoras $\cos^2 x = 1 - \sin^2 x$)

$$= \int (1-u^2) u^5 \, du$$

$$= \int u^5 - u^7 \, du = \frac{u^6}{6} - \frac{u^8}{8} + C$$

$$\int \cos^3 x \sin^5 x \, dx$$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$$

opg $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

$u = 1-x^2$

$u' = -2x$

$x = \frac{-1}{2} \cdot u'$

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$$= \int \frac{-1}{2} u' \cdot \frac{1}{\sqrt{u}}\,dx$$

$$= -\frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{-1}{2} \cdot \frac{u^{1/2}}{1/2} + C$$

$$= -\sqrt{u} + C = -\sqrt{1-x^2} + C$$

Lineær substitusjon

$u = ax+b$

$u' = a$

$$\int f(ax+b)\,dx$$

$$= \int \frac{1}{a} \cdot u' f(\overbrace{ax+b}^{u})\,dx$$

$$= \frac{1}{a}\int f(u)\,du$$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$
$$= F(b) - F(a)$$

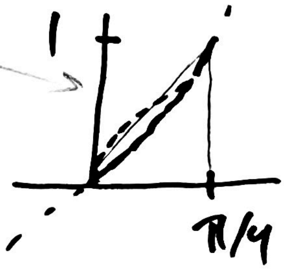
(6)

$$\int_a^b u'(x)\, f(u(x))\, dx$$
$$= F(u(x))\Big|_a^b$$
$$= F(u(b)) - F(u(a))$$
$$= \int_{u(a)}^{u(b)} f(u)\, du$$

NB grensene endres !

$\frac{1}{2}\ln 2 \sim 0.34657\ldots$ svaret virker rimelig

$\frac{1}{2}\,\pi/4 = \pi/8 \sim 0.392699$ areal trekant



$$\int_0^{\pi/4} \tan x\, dx$$

$$\int_0^{\pi/4} \frac{\sin x}{\cos x}\, dx$$

$w = \cos x$
$w' = -\sin x$

$$= \int_0^{\pi/4} \frac{-w'}{w}\, dx$$
$$= \int_{w(0)}^{w(\pi/4)} -\frac{1}{w}\, dw = -\ln|w|\Big|_1^{1/\sqrt{2}}$$
$$= -\ln\left(\frac{1}{\sqrt{2}}\right) + \ln 1 = \underline{\underline{\ln(\sqrt{2})}} = \frac{1}{2}\ln(2)$$

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$$\int \frac{\sin x}{\cos x}\,dx$$

$v = \sin x$

$v' = \cos x$

$\cos^2 = 1 - \sin^2 v$

$$= \int \underbrace{\cos x}_{v'} \cdot \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int v' \frac{v}{1-v^2}\,dx = \int \frac{v}{1-v^2}\,dv$$

$u = 1 - v^2$

$u' = -2v$

$v = \frac{-1}{2}u'$

$$= \int \frac{-1}{2}\frac{u'}{u}\,dv = \frac{-1}{2}\int \frac{1}{u}\,du$$

$$= \frac{-1}{2}\ln|u| + c = \frac{-1}{2}\ln|1-v^2| + c$$

$$= \frac{-1}{2}\ln|\underbrace{1-\sin^2 x}_{\cos^2 x}| + c$$

$$= \frac{-1}{2}\ln(|\cos x|^2) + c$$

$$= \underline{-\ln|\cos x| + c}$$

Dette fungerer men er unødig tungvint.

Det blir lettere med substitusjonen $u = \cos x$, $u' = -\sin x$

$$\int \frac{-u'}{u}\,dx = -\ln|u| + c = \underline{-\ln|\cos x| + c}$$

(8)

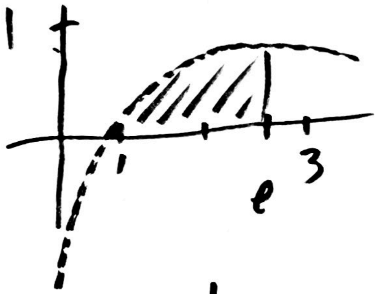
$$\int_1^e \frac{1}{x} \underbrace{\ln(x^2)}_{2\ln x} dx$$

$(\ln|x|)' = \frac{1}{x}$

$u = \ln x$

$u' = \frac{1}{x}$

$$= 2 \int_1^e \underbrace{\frac{1}{x}}_{u'} \underbrace{\ln x}_{u} dx$$



$$= 2 \int_{u(1)}^{u(e)} u \, du$$

$$= 2 \int_0^1 u \, du = 2 \left. \frac{u^2}{2} \right|_0^1$$

$$= 2\left(\frac{1}{2} - 0\right) = \underline{\underline{1}}$$

Alternativt:

$$\int \frac{1}{x} \ln(x^2) \, dx = 2 \int \underbrace{\frac{1}{x}}_{u'} \underbrace{\ln x}_{u} dx$$

$$= 2 \int u \, du = u^2 + C$$

$$= (\ln x)^2 + C$$

Så $\int_1^e \frac{1}{x} \ln(x^2) dx = (\ln x)^2 \Big|_1^e$

$$= (\ln e)^2 - (\ln 1)^2 = \underline{\underline{1}}$$

$$\int x(1+x^3)^7\,dx$$

(10)

$$\int \tfrac{1}{2}v'(1+v^{3/2})^7\,dx$$

$v = x^2$

$v' = 2x$

$$= \tfrac{1}{2}\int (1+v^{3/2})^7\,dv$$

$x = \tfrac{1}{2}v'$

vanskeligere enn ∫ vi starta med...

$x^3 = (x^2)^{3/2}$

$\parallel$

$|x|^3$

gyldig for $x>0$

$u = 1+x^3$

$u' = 3x^2$

$x = \dfrac{u'}{3x}$

$$\int x(1+x^3)^7\,dx$$

$$\int \frac{u'}{3x}\,u^7\,dx$$

$x^3 = u-1$

$x = \sqrt[3]{u-1}$

$$= \int \frac{1}{3}\cdot\frac{1}{\sqrt[3]{u-1}}\,u^7\cdot\underbrace{u'dx}_{du}$$

$$= \frac{1}{3}\int \frac{u^7}{\sqrt[3]{u-1}}\,du \quad \ldots$$

Kommer ikkje videre.

Her vil det fungere best å gange ut $x(1+x^3)^7$ og så integrere.

(11)

$$\int x \underbrace{\sqrt{x-2}}_{\sqrt{u}}\, dx$$

$u = x-2$

$u' = 1$

$x = u+2$

$$= \int \underbrace{u'}_{1} \underbrace{(u+2)}_{x} \underbrace{\sqrt{u}}_{\sqrt{x-2}}\, dx$$

$$= \int (u+2)\, u^{1/2}\, du$$

$$= \int u^{3/2} + 2u^{1/2}\, du$$

$$= \frac{u^{5/2}}{5/2} + 2\,\frac{u^{3/2}}{3/2} + c$$

$$= \underline{\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + c}$$