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Delvis integrasjon

Produktregelen $\quad (f(x)\,g(x))' = f'(x)\,g(x) + f(x)\cdot g'(x)$

$$(x\cos x)' = (x)'\cos x + x(\cos x)'$$
$$= \cos x - x\sin x$$

$$x\sin x = \underbrace{\cos x}_{(\sin x)'} - (x\cos x)'$$

$$x\sin x = (\sin x - x\cos x)'$$

$$\int x\sin x\,dx = \sin x - x\cos x + C$$

$$f \cdot g + c = \int f' \cdot g + f \cdot g' \, dx$$
$$= \int f' \cdot g \, dx + \int f \cdot g' \, dx$$

Delvis integras

$$\int f' \cdot g \, dx = f \cdot g - \int f \cdot g' \, dx$$

$$\int \underbrace{x}_{g} \underbrace{\sin x}_{f'} \, dx = x(-\cos x) - \int 1(-\cos x) \, dx$$

$f' = \sin x$
$f = -\cos x$

$$= -x\cos x + \int \cos x \, dx$$
$$= -\underline{x \cos x + \sin x} + C$$

$$\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' \, dx$$

opg

$$\int \underbrace{3x}_{} e^x \, dx = 3 \int \underbrace{x}_{v} \underbrace{e^x}_{u'} dx$$

$u' = e^x$
$u = e^x$
$v = x$
$v' = 1$

$$= 3\left(x e^x - \int 1 \cdot e^x \, dx \right)$$

$$= 3(x e^x - e^x) = 3e^x(x-1) + c$$

$$= \underline{3(x-1)e^x + c}$$

$u' = e^{2x-1}$
$u = \frac{1}{2} e^{2x-1}$

$$\int \underbrace{(3x+1)}_{v} \underbrace{e^{2x-1}}_{u'} dx$$

$$= (3x+1) \cdot \frac{1}{2} e^{2x-1} - \int 3 \cdot \frac{1}{2} e^{2x-1} dx$$

$$= (3x+1) \frac{1}{2} e^{2x-1} - \frac{3}{2} \cdot \frac{1}{2} e^{2x-1} + c$$

$$= \underline{\frac{1}{2}\left(3x - \frac{1}{2}\right) e^{2x-1} + c}$$

$$\int \underbrace{x^2}_{u'} \cdot \underbrace{\ln|x|}_{v} \, dx = \frac{x^3}{3}\ln|x| - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx$$

$u' = x^2$

$u = \frac{x^3}{3}$

$v = \ln|x|$

$v' = \frac{1}{x}$

$$= \frac{x^3}{3}\ln|x| - \frac{1}{3}\int x^2 \, dx$$

$$= \frac{x^3}{3}\ln|x| - \frac{1}{3} \cdot \frac{1}{3} x^3 + c$$

$$= \underline{\frac{x^3}{9}\left(3\ln|x| - 1\right) + c}$$

$$\int \ln|x|\,dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln|x|}_{v}\,dx \qquad u' = 1,\quad u = x$$

$$= x\ln|x| - \int \underbrace{x \cdot \frac{1}{x}}_{1}\,dx$$

$$= x\ln|x| - x + C$$

$$= \underline{x(\ln|x| - 1) + C}$$

$$\int \underbrace{x^2}_{v}\underbrace{e^x}_{u'}\,dx = x^2 e^x - \int 2x e^x\,dx$$

$$= x^2 e^x - 2\underbrace{\int x e^x\,dx}_{(x-1)e^x} + C$$

$$= \underline{(x^2 - 2x + 2)e^x + C}$$

$$\int \underbrace{x^3}_{v} \underbrace{e^x}_{u'} dx = u \cdot v - \int v' \cdot u \, dx$$

$$= x^3 e^x - \int 3x^2 \cdot e^x dx$$

$$= x^3 e^x - 3(x^2 - 2x + 2)e^x + c$$

$$= \underline{(x^3 - 3x^2 + 6x - 6)e^x + c}$$

$$\int \underbrace{x^n}_{v} \underbrace{e^x}_{u'} dx = x^n e^x - \int n x^{n-1} e^x dx$$

Rekursive formel $\boxed{\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx}$

Opg Rekursiv formel for $\int \underbrace{(ax+b)^n}_{v} \underbrace{e^{c \cdot x}}_{u'} dx$

$$\int (ax+b)^n e^{cx}\,dx = \frac{1}{c}(ax+b)^n e^{cx} - \int a\cdot n\,(ax+b)^{n-1}\cdot\frac{1}{c}e^{cx}\,dx \quad \Bigg| \begin{array}{l} u' = e^{cx} \\ u = \frac{1}{c}e^{cx} \end{array}$$

$$\int (ax+b)^n e^{cx}\,dx = \frac{1}{c}(ax+b)^n e^{cx} - \frac{a\cdot n}{c}\int (ax+b)^{n-1} e^{cx}\,dx$$

Kombinasjon av substitusjon og delvis integrasjon

$$\int x^3 e^{x^2}\,dx \qquad \begin{array}{l} u = x^2 \\ u' = 2x \end{array}$$

$$\int \underbrace{2x}_{u'}\cdot\underbrace{\frac{1}{2}x^2}_{\frac{1}{2}\cdot u}\;\underbrace{e^{x^2}}_{e^u}\,dx \qquad u'dx = \frac{du}{dx}dx \sim du$$

$$= \int \frac{1}{2}u\,e^u\,du = \frac{1}{2}(u-1)e^u + C$$

$$= \underline{\frac{1}{2}(x^2-1)e^{x^2} + C}$$

$$\int \sqrt{x}\, e^{\sqrt{x}}\, dx$$

$$u = \sqrt{x} = x^{1/2}$$

$$u' = \frac{1}{2\sqrt{x}}$$

$$= \int \underbrace{\frac{1}{2\sqrt{x}}}_{u'} \cdot \underbrace{2\sqrt{x}\cdot\sqrt{x}}_{2u\cdot u}\, \underbrace{e^{\sqrt{x}}}_{e^{u}}\, dx$$

$$= 2\int u^2 e^u\, du = 2(u^2 - 2u + 2)e^u + c$$

$$\int \sqrt{x}\, e^{\sqrt{x}} = \underline{2(x - 2\sqrt{x} + 2)e^{\sqrt{x}} + c}$$

Førsøk å finne $\int \sqrt{1-x^2}\, dx = \int \underset{u'}{1} \cdot \underbrace{\sqrt{1-x^2}}_{v}\, dx \quad \ldots$

$$u' = e^{ax}$$
$$u = \frac{1}{a} e^{ax}$$

$$\int \underbrace{e^{ax}}_{u'} \underbrace{\sin(bx)}_{v} \, dx$$

$$= \frac{1}{a} e^{ax} \sin(bx) - \int \underbrace{\frac{1}{a} e^{ax}}_{w'} \underbrace{b \cos(bx)}_{z} \, dx$$

$$w' = \frac{1}{a} e^{ax}$$
$$w = \frac{1}{a^2} e^{ax}$$

$$= \frac{1}{a} e^{ax} \sin(bx) - \left(\frac{1}{a^2} e^{ax} \cdot b \cos(bx) - \int \frac{1}{a^2} e^{ax} \cdot b^2 (-\sin(bx)) \, dx \right)$$

$$= \frac{1}{a} e^{ax} \sin(bx) - \frac{1}{a^2} e^{ax} b \cos(bx) - \frac{b^2}{a^2} \int e^{ax} \sin(bx) \, dx$$

$$\left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin(bx) \, dx = \frac{e^{ax}}{a^2} \left(a \sin(bx) - b \cos(bx) \right) + c$$

$$\int e^{ax} \sin(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} \left(a \sin(bx) - b \cos(bx) \right) + c$$

$$\int e^{2x} \sin(3x)\, dx = \frac{e^{2x}}{13}\left(2\sin(3x) - 3\cos(3x)\right) + c$$

$$\int_0^{\pi/2} \underbrace{x}_{v} \cdot \underbrace{\cos(x)}_{u'}\, dx = \left(x \sin x - \int 1 \sin x\, dx\right)\Big|_0^{\pi/2}$$

$$x \sin x + \cos x \Big|_0^{\pi/2}$$

$$= x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\, dx$$

$$= \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) - 0 \qquad \cos\frac{\pi}{2} - \cos 0$$

$$= \frac{\pi}{2} - 1 \sim \underline{0{,}57}$$

$$\int_a^b u' v\, dx = u \cdot v \Big|_a^b - \int_a^b u \cdot v'\, dx$$

$$\int \frac{1}{x^2-1}\,dx \quad ?$$

Delbrøksoppspalting

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

Vi ser mer på slike eksempler neste uke.

Øving fredag

$$\int \sqrt{1-x^2}\,dx = \int 1\cdot\sqrt{1-x^2}\,dx$$

$$= x\sqrt{1-x^2} - \int x \frac{-x}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$\left(\sqrt{1-x^2}\right)' = \frac{1}{2}\frac{-2x}{(1-x^2)^{1/2}} = \frac{-x}{\sqrt{1-x^2}}$$

vi skriver om $\frac{-x^2}{\sqrt{1-x^2}}$

som $\frac{1-x^2-1}{\sqrt{1-x^2}}$

$$= \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}$$

$$\int\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\,dx$$

$$2\int\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin(x)\right) + c$$

16.123 b) $\int \frac{1}{x \ln x}\, dx$ $\qquad (\ln x)' = \frac{1}{x}$

$\int \underbrace{\frac{1}{x}}_{u'} \cdot \frac{1}{\ln x}\, dx$

$\int u' \cdot \frac{1}{u}\, dx$

$= \int \frac{1}{u}\, du = \ln|u| + C$

$= \underline{\ln|\ln x| + C}$

$u = \cos x$

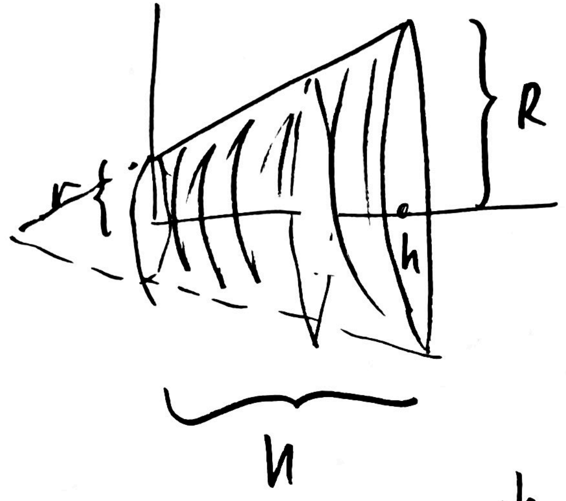
$u' = -\sin x$

d) $\int \frac{\tan x}{\cos x}\, dx = \int \frac{\sin x}{\cos^2 x}\, dx$

$= \int -\frac{u'}{u^2}\, dx$

$= -\int \frac{du}{u^2} = \frac{1}{u} + C = \underline{\frac{1}{\cos x} + C}$

16.117



$$f = \frac{R-r}{h}x + r$$

$$V = \int_0^h \pi f_{(x)}^2 \, dx = \pi \int_0^h \left(\frac{R-r}{h}x + r\right)^2 dx$$

$$U = \frac{R-r}{h}x + r$$

$$U' = \frac{R-r}{h}$$

$$dx = \frac{h}{R-r} du$$

$$= \pi \int_r^R \frac{h}{R-r} U^2 \, du$$

$$= \frac{\pi}{3}\frac{h}{R-r} \; U^3 \Big|_r^R$$

$$\underline{V = \frac{\pi}{3}\frac{h}{R-r}\left(R^3 - r^3\right)} = \underline{\frac{\pi}{3}h\left(R^2 + Rr + r^2\right)}$$

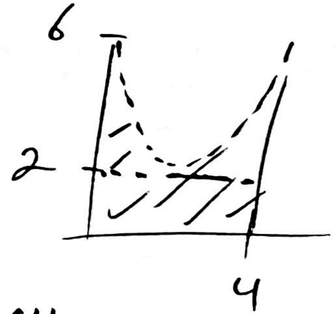
$$(x^n - 1) = (1 + x + x^2 + \cdots + x^{n-1})(x-1)$$

$x = \frac{b}{a}$: $$\left(\frac{b}{a}\right)^n - 1 = \left(1 + \frac{b}{a} + \left(\frac{b}{a}\right)^2 + \cdots + \left(\frac{b}{a}\right)^{n-1}\right)\left(\frac{b}{a} - 1\right)$$

ganger med a^n på beggesider av likhetstegnet

$$\underline{b^n - a^n = (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots b^{n-1})(b-a)}$$

$n=2$ er konjugatsetningen.



16.112 $f(x) = x^2 - 4x + 6 = (x-2)^2 + 2$

a) $A = \int_0^4 x^2 - 4x + 6 \, dx$

$$= \int_0^4 (x-2)^2 + 2 \, dx = \int_0^4 (x-2)^2 dx + \int_0^4 2 \, dx$$

$$= \int_{-2}^{2} u^2 du + 2 \cdot 4 = \frac{u^3}{3}\Big|_{-2}^{2} + 8$$

$$= 2\frac{8}{3} + 8 = 8\left(\frac{2}{3} + 1\right) = \frac{8 \cdot 5}{3} = \underline{\frac{40}{3}}$$

13.333...

b) $V = \int_0^4 \pi f(x)^2 \, dx = \pi \int_0^4 ((x-2)^2+2)^2 \, dx$

$= \pi \int_{-2}^{2} (u^2+2)^2 \, du$

$= \pi \int_{-2}^{2} u^4 + 4u^2 + 4 \, du$

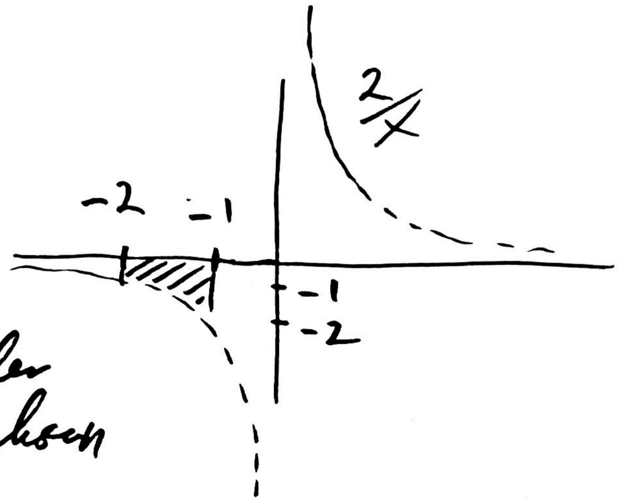
$= \pi \left[\frac{u^5}{5} + \frac{4}{3}u^3 + 4u \right]_{-2}^{2}$

$= 2\pi \left[\frac{2^5}{5} + \frac{4}{3}2^3 + 4 \cdot 2 \right]$

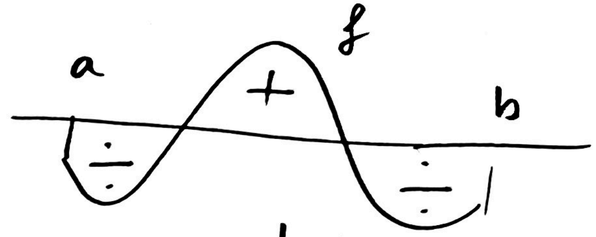
$= 2\pi \left[\frac{32}{5} + \frac{32}{3} + 8 \right] = 16\pi \left[\frac{4}{5} + \frac{4}{3} + 1 \right]$

$16\pi \left[3.1333\ldots \right]$

15.66



under x-aksen



$\int_a^b f(x)\,dx$

$$A = -\int_{-2}^{-1} \frac{2}{x}\,dx = -2\ln|x| \Big|_{-2}^{-1}$$

$$= -2(\underbrace{\ln(1)}_{0} - \ln(2))$$

$$= 2\ln(2) \sim \underline{1{,}38629}$$