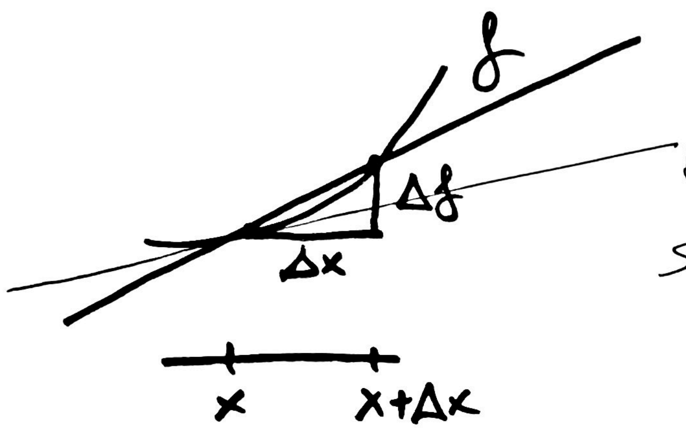
12.10

Derivasjon

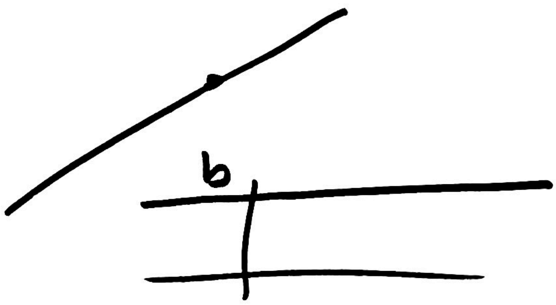
$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



momentan vekstfart

tangentlinje $(x, f(x))$
stigningstallet er $f'(x)$.

$$(ax+b)' = a$$



$$(b)' = 0$$

$$(x^2)' = 2x \quad \text{(tidligere)}$$

$$(x^n)' = n x^{n-1}$$

n nat tall
(n reelt tall)

Derivasjon er lineær:

$$(k \cdot f(x))' = k \cdot f'(x)$$

Vi viser dette:

$$(kf(x))' = \lim_{\Delta x \to 0} \frac{kf(x+\Delta x) - kf(x)}{\Delta x}$$

$$= k \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= k f'(x).$$

$$(f(x)+g(x))' = f'(x) + g'(x)$$

bevis: som ovenfor, benytt grensesetningene.

ex

$$(3x^2-3)' = (3x^2)' + (-3)'$$

$$= 3\underbrace{(x^2)'}_{2x} + \underbrace{(-3)'}_{0}$$

$$= 3 \cdot 2x = \underline{6x}$$

$a \neq 0$ (opplagt for $a=0$)

Linear kjerneregel

$$(f(ax+b))' = a f'(ax+b)$$

$$= \lim_{\Delta x \to 0} a \cdot \frac{f(a(x+\Delta x)+b) - f(ax+b)}{a \cdot \Delta x}$$

$$= a \lim_{\Delta x \to 0} \frac{f(ax+b+(a\Delta x)) - f(ax+b)}{a\Delta x}$$

$$= a f'(ax+b).$$

Ex

$$((2+3x)^5)' = 5(2+3x)^4 \cdot 3$$

$$= \underline{15(2+3x)^4}$$

$$\left(\sqrt{x}\right)' = \left(x^{1/2}\right)' = \frac{1}{2\sqrt{x}} \qquad \left(\begin{matrix}\text{vist}\\ \text{mandag}\end{matrix}\right)$$

$$\left(\frac{1}{2}x^{\frac{1}{2}-1} \ldots\right)$$

$$\left(\sqrt{3-2x}\right)' = \frac{1}{2\sqrt{3-2x}}\cdot(-2)$$

(benytter lineær kjerneregel)

$$= \frac{-1}{\sqrt{3-2x}}$$

$$\left(\sqrt[4]{10+5x}\right)' = \left(\left(10+5x\right)^{1/4}\right)'$$

$$= \frac{1}{4}\cdot(10+5x)^{\frac{1}{4}-1}\cdot(10+5x)'$$

$$= \frac{5}{4}\,(10+5x)^{-3/4} = \frac{5}{4}\left(\left(10+5x\right)^{3/4}\right)^{-1}$$

$$= \frac{5}{4}\,\frac{1}{(10+5x)^{3/4}} = \frac{5}{4\sqrt[4]{(10+5x)^3}}$$

$$= \underline{\frac{5}{4\left(\sqrt[4]{10+5x}\right)^3}}$$

$$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$$

$$(x^{-1})' = -1 \cdot x^{-1-1} = -1 \cdot x^{-2} = \frac{-1}{x^2}$$

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)}$$ vi viser dette:

Fra definisjonen:

$$\left(\frac{1}{f(x)}\right)' = \lim_{\Delta x \to 0} \underbrace{\left(\frac{1}{f(x+\Delta x)} - \frac{1}{f(x)}\right)}_{\frac{f(x)}{f(x)\cdot f(x+\Delta x)} - \frac{f(x+\Delta x)}{f(x)\, f(x+\Delta x)}} \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x) - f(x+\Delta x)}{\Delta x \cdot f(x)\, f(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{f(x)\, f(x+\Delta x)} \cdot \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

grense-setning $= \lim_{\Delta x \to 0} \frac{-1}{f(x)\, f(x+\Delta x)} \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \frac{-1}{(f(x))^2} \cdot f'(x)$$

vi har benyttet:

$$f(x) - f(x+\Delta x) = -1 \cdot (f(x+\Delta x) - f(x))$$

Viser $(x^n)' = n x^{n-1}$ n heltall.

$$y^n - x^n = (y-x)(y^{n-1} + xy^{n-2} + \dots + x^{n-2} \cdot y + x^{n-1}) \quad n \in \mathbb{N}$$

likhet
"identitet"

n ledd

Bevis: Ganger ut parentesene ...

$$y^n + xy^{n-1} + x^2 y^{n-2} + \dots + x^{n-1} \cdot y$$
$$-(\quad xy^{n-1} + x^2 y^{n-2} + \dots + x^{n-1} \cdot y + x^n)$$

nesten alle leddene kanseleres.

$$= y^n - x^n.$$

eks: konjugatsetningen $n=2$ en tilfellet

$n=3$ $\quad y^3 - x^3 = (y-x)(y^2 + xy + x^2)$

$n=4$ $\quad y^4 - x^4 = (y-x)(y^3 + y^2 x + yx^2 + x^3)$

$$(x^n)' = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

la $y = x + \Delta x$

$$= \lim_{\Delta x \to 0} \frac{\overbrace{\Delta x}^{(x+\Delta x)-x}}{\Delta x} \Big(\underbrace{(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2} \cdot x + \dots x^{n-1}}_{n \text{ ledd}} \Big)$$

$$= \underbrace{x^{n-1} + \dots x^{n-1}}_{n \text{ ledd}} = \underline{n \cdot x^{n-1}}$$

n naturlig tall.

$n \in \mathbb{N}$ $\qquad \frac{1}{x^n} = x^{-n}$

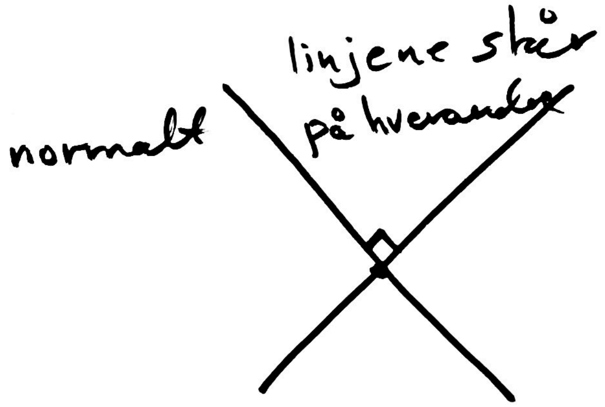
$$\left(\frac{1}{x^n}\right)' = \frac{-1}{(x^n)^2} \cdot \underbrace{(x^n)'}_{n \cdot x^{n-1}}$$

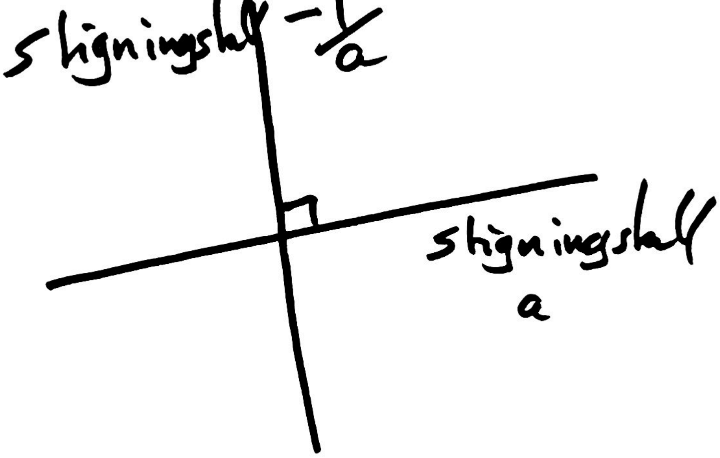
$$= \frac{-n\, x^{n-1}}{x^{2n}} = -n\, x^{n-1} \cdot x^{-2n}$$

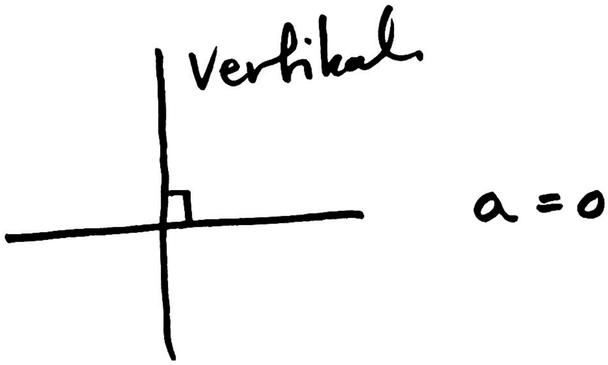
$$= \underline{(-n)\, x^{-n-1}}$$

Derfor er $\qquad (x^m)' = m\, x^{m-1}$ for alle $m \in \mathbb{Z}$.

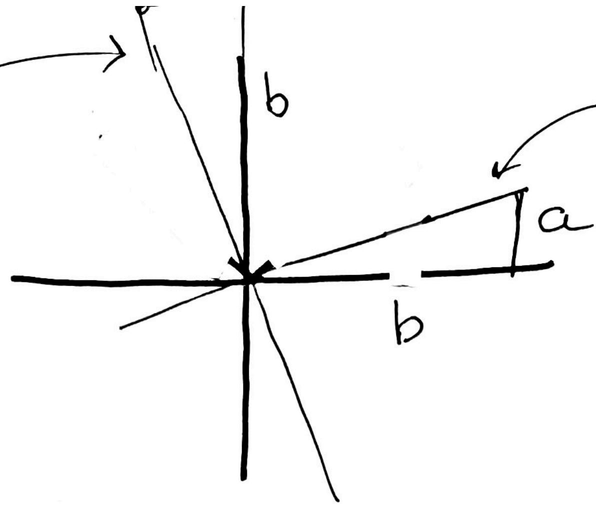
Tangentlinjer og normallinjer







stigningstall $\frac{b}{-a} = -\frac{1}{(a/b)}$



stigningstall $\frac{a}{b}$

Finn tangent og normallinje til

$f = x^2$ i $(1, 1)$.

$f'(x) = 2x$.

stigningstallet til tangentlinjen i $x = 1$ er $2 \cdot 1 = 2$.

$y = 2x + b$ gå gjennom $(1, 1)$

gir $b = -1$

$\underline{y = 2x - 1}$ tangentlinjen

(generelt: $y = f'(a)(x - a) + f(a)$)

Normallinjen har stigningstall $\frac{-1}{2}$

$y = -\frac{1}{2}x + b$ gjennom $(1, 1)$

gir $1 = -\frac{1}{2} \cdot 1 + b$ så $b = 3/2$.

$y = \frac{-x}{2} + \frac{3}{2}$ normallinjen.

generelt: normallinjen til $f(x)$ i $(a, f(a))$ er

$f'(a) \neq 0$ $$y = \frac{-1}{f'(a)}(x-a) + f(a)$$

hvis $f'(a) = 0$ så er normalen vertikal: $x = a$

$$\boxed{\text{Hvis } f(x) \text{ er deriverbar i } a \text{ så er den også kontinuerlig.}}$$

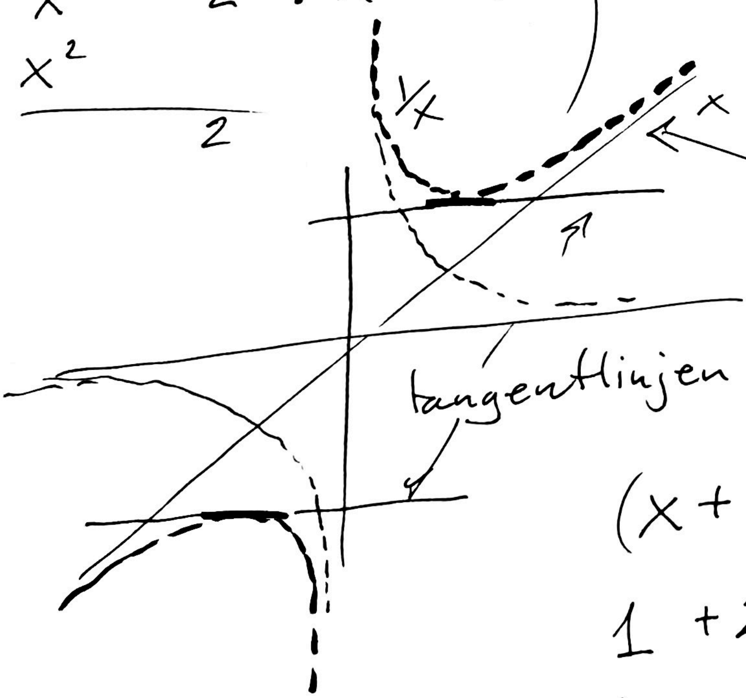
f kont $\Leftrightarrow \lim_{\Delta x \to 0} \Delta f = 0$, f deriverbar $\lim_{\Delta x \to 0} \Delta f = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \cdot \Delta x$

$= f'(a) \lim_{\Delta x \to 0} \Delta x$

$= 0$

oppg (8 i oblig 2) skisser grafen til $\frac{x^2+2}{x}$ og finn topp og bunnpunkt.

$$\left(\begin{array}{l} x^2 \quad 2 : x = x \\ \underline{x^2} \\ \quad\quad 2 \end{array} \right) \qquad \frac{x^2+2}{x} = x + \frac{2}{x}$$



skrå asymptote.

tangentlinjen er horisontal.

$$\left(x + \frac{2}{x}\right)' = (x)' + 2\left(\frac{1}{x}\right)'$$

$$1 + 2\,\frac{-1}{x^2} = 1 - \frac{2}{x^2}.$$

Den deriverte er lik 0: $1 - \frac{2}{x^2} = 0$

$x^2 = 2$ som gir $x = \pm\sqrt{2}$.

Bunnpunkt i $\left(\sqrt{2}, \frac{4}{\sqrt{2}}\right) = \underline{(\sqrt{2}, 2\sqrt{2})}$

Toppunkt i $\underline{(-\sqrt{2}, -2\sqrt{2})}$.

Oppg (10 oblig) Bestem a og b

slik at $f(x) = \begin{cases} x^2 & x \leq 2 \\ ax+b & x > 2 \end{cases}$

blir deriverbar for alle x.

$f(x)$ kont. og deriverbar for $x \neq 2$

$$f'(x) = \begin{cases} 2x & x < 2 \\ a & x > 2 \end{cases}$$

kontinuerlig i 2: $\lim_{x \to 2^-} f(x) = 4 = f(2)$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax+b = \underline{2a+b}$$

kont. $\Leftrightarrow$ $\underline{2a+b = 4}$.

Deriverbar i $x = 2$ hvis kont.

og $\lim_{x \to 2^-} (2x) = \lim_{x \to 2^+} a$

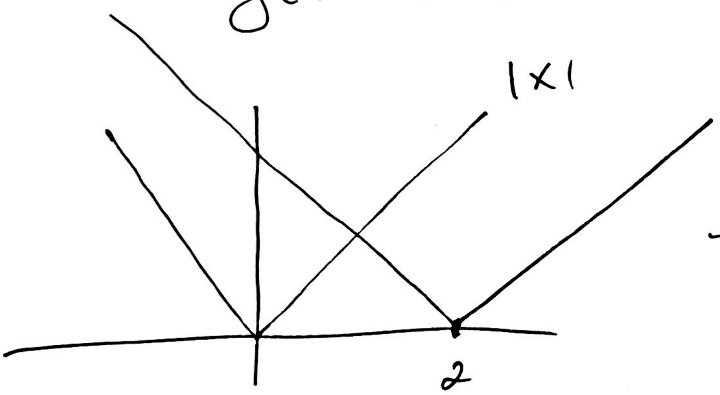
$$2a + b = 4$$
$$4 = a$$

Dette gir $\quad \underline{a = 4}$

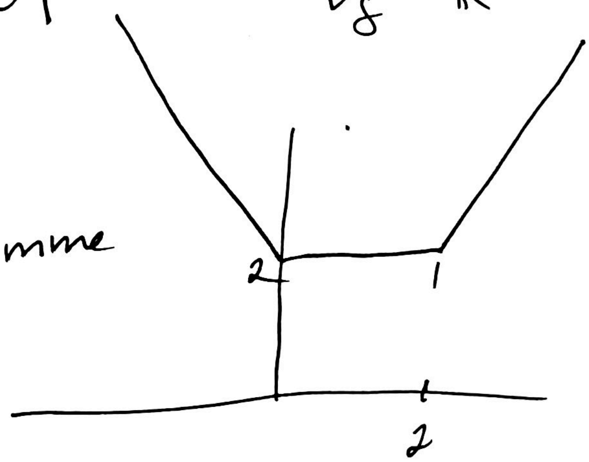
$\qquad\qquad\quad b = 4 - 2a = \underline{-4}$

Oppg. Skisser grafen til

$$f(x) = |x| + |x-2| \qquad D_f = \mathbb{R}$$



Summe



$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} + \begin{cases} x-2 & x \geq 2 \\ 2-x & x < 2 \end{cases}$$

$$= \begin{cases} 2x - 2 & x \geq 2 \\ 2 & 0 \leq x < 2 \\ 2-2x & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2 & x > 2 \\ 0 & 0 < x < 2 \\ -2 & x < 0 \end{cases}$$

ikke deriverbar i $x = 0, 2$.