17.2.23

14.5 Vektorproduktet (Kryssproduktet)

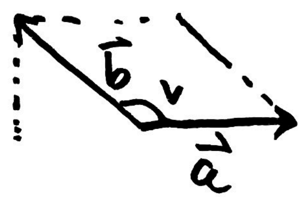
Definisjon

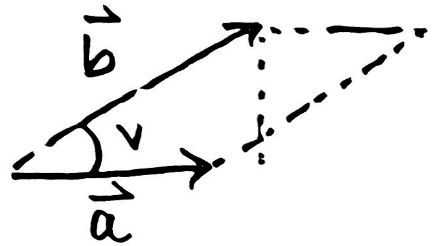
$\vec{a}, \vec{b}$ i $\mathbb{R}^3$

$\vec{a} \times \vec{b}$ er en vektor i $\mathbb{R}^3$

størrelsen: $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(v)$

arealet til parallellogrammet utspent av $\vec{a}$ og $\vec{b}$





retningen: $\vec{a} \times \vec{b}$ vinkelrett på $\vec{a}$ og $\vec{b}$

$\vec{a}, \vec{b}$ og $\vec{a} \times \vec{b}$ er et høyrehånd-system.

$$\vec{a} \times \vec{a} = \vec{0}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$ antisymmetrisk.

Resultat: Vektorproduktet er bilineært.

$$(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b})$$

$$(\vec{a}_1 + \vec{a}_2) \times \vec{b} = \vec{a}_1 \times \vec{b} + \vec{a}_2 \times \vec{b}$$

eks:

$$(2\vec{a} - 3\vec{b}) \times (4\vec{a} + 5\vec{b})$$

$$= \underbrace{2\vec{a} \times 4\vec{a}}_{\vec{0}} + 2\vec{a} \times 5\vec{b} - 3\vec{b} \times 4\vec{a} - \underbrace{3\vec{b} \times 5\vec{b}}_{\vec{0}}$$

$$= 10\vec{a} \times \vec{b} - 12\underbrace{\vec{b} \times \vec{a}}_{-\vec{a} \times \vec{b}} = 10\vec{a} \times \vec{b} + 12\vec{a} \times \vec{b}$$

$$= \underline{22\,\vec{a} \times \vec{b}}$$

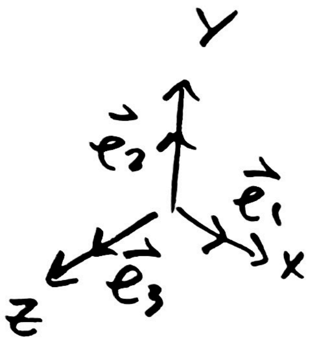
$$[x_1, y_1, 0] \times [x_2, y_2, 0]$$

lineært produkt

$$= (x_1\vec{e}_1 + y_1\vec{e}_2) \times (x_2\vec{e}_1 + y_2\vec{e}_2)$$

$$= x_1x_2 \underbrace{\vec{e}_1 \times \vec{e}_1}_{\vec{0}} + y_1x_2 \underbrace{\vec{e}_2 \times \vec{e}_1}_{-\vec{e}_1 \times \vec{e}_2} + x_1y_2\,\vec{e}_1 \times \vec{e}_2 + y_1y_2 \underbrace{\vec{e}_2 \times \vec{e}_2}_{\vec{0}}$$

$$= (x_1y_2 - x_2y_1)\underbrace{\vec{e}_1 \times \vec{e}_2}_{\vec{e}_3}$$



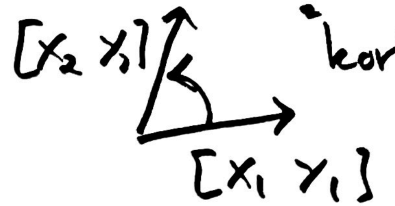
$$= \left[0,\ 0,\ \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right]$$

(determinant
$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$)

Dette viser at $\text{abs}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ = arealet av parallellogrammet utspent av $[x_1, y_1]$ og $[x_2, y_2]$.

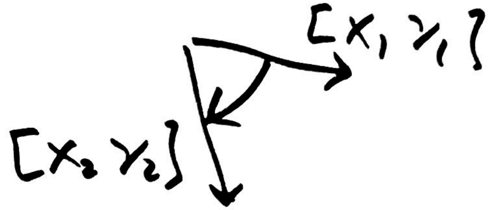
Fortegn $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

positivt

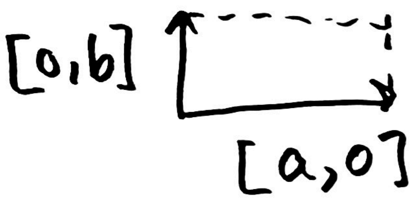


"korteste veg"
+ retni..

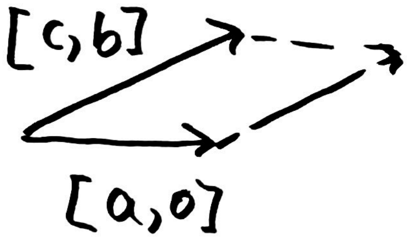
negativt



÷ retning



$$[a,0] \times [0,b] = \left[0, 0, \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}\right] = \underline{[0, 0, a \cdot b]}$$



$$[a,0] \times [c\ b] = \left[0, 0, \begin{vmatrix} a & 0 \\ c & b \end{vmatrix}\right] = [0, 0, a \cdot b - 0 \cdot c] = \underline{[0, 0, a \cdot b]}$$

Kryssprodukt på koordinatform

$$\overset{\vec{v}_1}{[x_1, y_1, z_1]} \times \overset{\vec{v}_2}{[x_2, y_2, z_2]}$$

× er lineært

$$= (x_1\vec{e}_1 + y_1\vec{e}_2 + z_1\vec{e}_3) \times (x_2\vec{e}_1 + y_2\vec{e}_2 + z_2\vec{e}_3)$$

$$= \vec{0} + \vec{0} + \vec{0} + x_1 y_2\, \vec{e}_1 \times \vec{e}_2 + y_1 x_2\, \vec{e}_2 \times \vec{e}_1$$
$$x_1 z_2\, \vec{e}_1 \times \vec{e}_3 + x_2 z_1\, \vec{e}_3 \times \vec{e}_1$$
$$y_1 z_2\, \vec{e}_2 \times \vec{e}_3 + y_2 z_1\, \vec{e}_3 \times \vec{e}_2$$

(Benytter: $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$ $\quad \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$, $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$)

$$\vec{v}_1 \times \vec{v}_2 = (x_1 y_2 - y_1 x_2)\overbrace{\vec{e}_1 \times \vec{e}_2}^{\vec{e}_3} + (x_1 z_2 - z_1 x_2)\overbrace{\vec{e}_1 \times \vec{e}_3}^{-\vec{e}_2}$$
$$(y_1 z_2 - z_1 y_2)\underbrace{\vec{e}_2 \times \vec{e}_3}_{\vec{e}_1}$$

$$[x_1, y_1, z_1] \times [x_2, y_2, z_2]$$

$$= \left[\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \; - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \; \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right]$$

$$= \det \begin{bmatrix} \overset{+}{\vec{e}_1} & \overset{-}{\vec{e}_2} & \overset{+}{\vec{e}_3} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

Eks: $\vec{a} = [1, 2, 3]$
$\vec{b} = [4, 5, 6]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \vec{e}_1 - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \vec{e}_2 + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \vec{e}_3$$

$$= [(2 \cdot 6 - 3 \cdot 5), \; -(1 \cdot 6 - 3 \cdot 4), \; (1 \cdot 5 - 2 \cdot 4)]$$

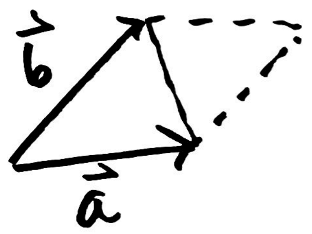
$$\vec{a} \times \vec{b} = [-3, 6, -3] = \underline{3[-1, 2, -1]}$$

$(\vec{a}\times\vec{b})\cdot\vec{a} = 3[-1,2,-1]\cdot[1,2,3] = 0$ ✓

$(\vec{a}\times\vec{b})\cdot\vec{b} = 3[-1,2,-1]\cdot[4,5,6] = 0$ ✓

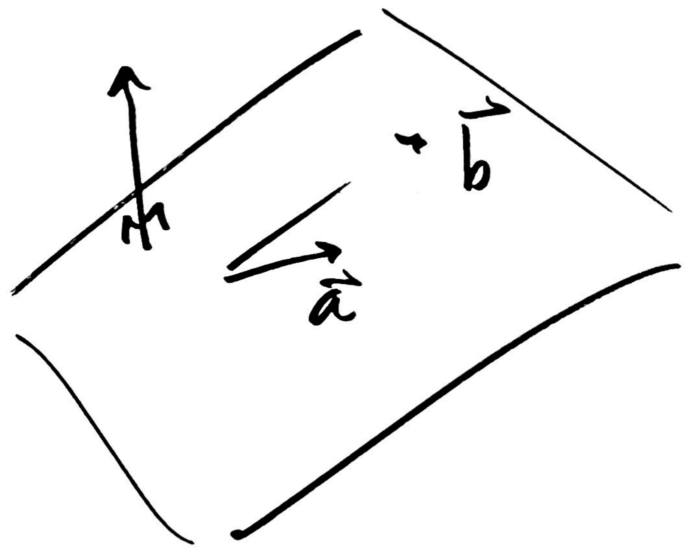
Arealet til parallellogrammet utspent av $\vec{a}$ og $\vec{b}$

er $|\vec{a}\times\vec{b}| = 3|[-1,2,-1]| = \underline{3\sqrt{6}}$



Arealet til trekanten utspent av $\vec{a}$ og $\vec{b}$

er $\frac{1}{2}|\vec{a}\times\vec{b}| = \underline{\frac{3}{2}\sqrt{6}}$



$\vec{a},\vec{b}$ utspenner et plan.

$\vec{a}\times\vec{b}$ og derfor

$[1,-2,1]$ er normalvektor

til planet.

$A(1,-2,4)$ $\quad$ $B(-2,-3,1)$ $\quad$ $O(0,0,0)$

opg

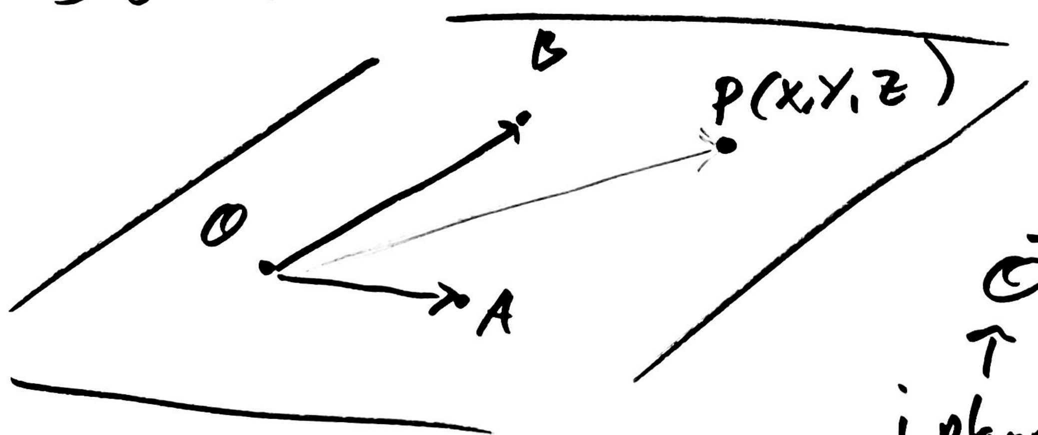
Finn: $\overrightarrow{OA} \times \overrightarrow{OB}$ $\quad$ $\overrightarrow{OA} = [1,-2,4]$, $\overrightarrow{OB} = [-2,-3,1]$.

$$= \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -2 & 4 \\ -2 & -3 & 1 \end{vmatrix} = \left[\begin{vmatrix} -2 & 4 \\ -3 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ -2 & -3 \end{vmatrix} \right]$$

$$= [-2-(-12), -(1-(-8)), -3-(-2)^2]$$

$$= \underline{[10, -9, -7]}$$

Likning for planet gjennom O, A og B.



$P(x,y,z)$ ligger i planet $\Leftrightarrow$

$\underbrace{\overrightarrow{OP}}_{\text{i planet}} \perp$ normalvektor til planet.

$$\underbrace{[x, y, z]}_{\overrightarrow{OP}} \bullet \underbrace{[10, -9, -7]}_{\overrightarrow{OA} \times \overrightarrow{OB}} = 0$$

$$10x - 9y - 7z = 0$$

(x, y, z) ligger i planet hvis og bare hvis likningen er oppfylt.

Beskriv planet: $x - y + 2z = 5$

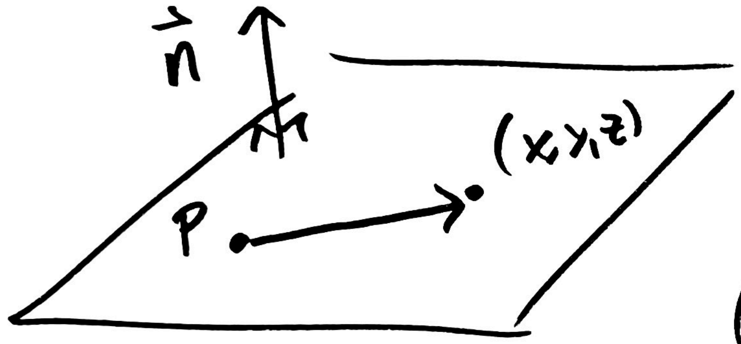
En normalvektor er $\vec{n} = [1, -1, 2]$

Et punkt i planet er: $(1, 0, 2)$
(oppfyller likningen)

Plan med normalvektor $\vec{n}$ gjennom punkt P:

$$[x, y, z] \cdot \vec{n} = \vec{OP} \cdot \vec{n}$$

$$\underline{ax + by + cz = d} \qquad [a, b, c] \neq \vec{0}$$

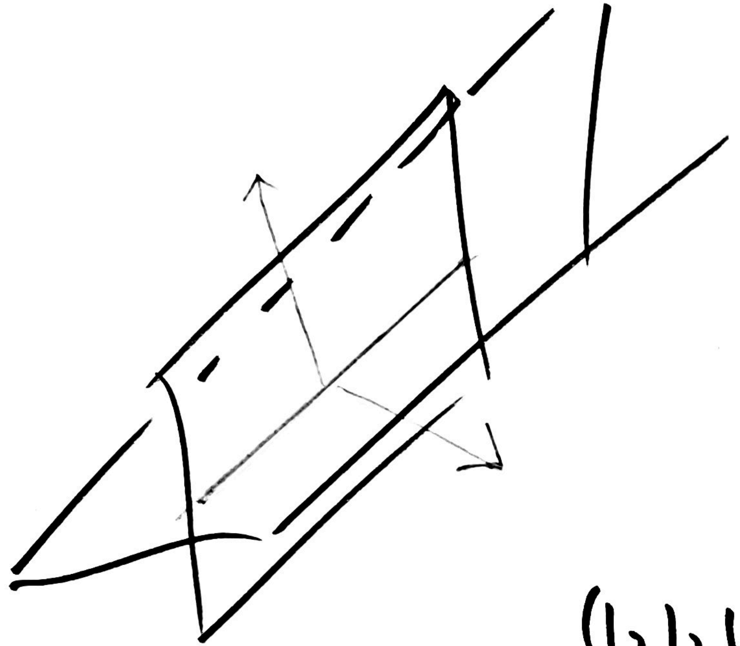


(x, y, z) i planet $\Leftrightarrow$

$$\overrightarrow{P(x,y,z)} \cdot \vec{n} = 0$$

$$\left(\overrightarrow{O(x,y,z)} - \vec{OP}\right) \cdot \vec{n} = 0$$

$$\underline{[x, y, z] \cdot \vec{n} = \vec{OP} \cdot \vec{n}}$$



To plan snitter typisk i en linje.

$$x + 2y + 3z = 6$$

$$4x + 5y + 6z = 15$$

$(1,1,1)$ er med i begge planene.

$\vec{n}_1 = [1,2,3]$ er normalvektor til plan 1

$\vec{n}_2 = [4,5,6]$ ——————————— 2

Snittlinjen må stå normalt på både $\vec{n}_1$ og $\vec{n}_2$

en retningsvektor er gitt ved

$$\vec{n}_1 \times \vec{n}_2 = 3[-1,2,-1]$$

velge Retningsvektor $\vec{r} = [1,-2,1]$

Linjen er parametrisert av $\underline{[x,y,z] = [1,1,1] + t[1,-2,1]}$