8 april 25

Fundamentalteoremet i kalkulus

15D

$f(x)$

Antideivert $F(x)$ hvis $F'(x) = f(x)$

Flere mulige antideiverte: Bestemt opp til å legge til en konstant (på hver komponent av D_f)

$\int f(x)\,dx$ = mengden av antideiverte til $f(x)$

Ubestemt integral

$= F(x) + C$

↑ en antideivert.

$\int f(x)\,dx$ lineær: $f(x)$. Dvs. respekterer skalarmultiplikasjon addisjon.

* $\int 3x + \frac{4}{x}\,dx = 3\int x\,dx + 4\int \frac{1}{x}\,dx$

$= 3\frac{x^2}{2} + 4\ln|x| + C$

En antideivert: $\begin{cases} 3x^2/2 + 4\ln|x| + 7 & x<0 \\ 3x^2/2 + 4\ln|x| + 11 & x>0. \end{cases}$

* $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \qquad r \neq -1.$

$\int \frac{7}{\sqrt{x}}\,dx = 7\int \frac{1}{x^{1/2}}\,dx = 7\int x^{-1/2}\,dx$

$= 7\frac{x^{1/2}}{1/2} + C = \underline{14\sqrt{x} + C}$

Fundamentalteoremet

$$I(x) = \int_a^x f(t)\,dt$$

Hvis $f(x)$ er kontinuerlig, da

$$\frac{d}{dx} I(x) = f(x).$$

(DVS. $I(x)$ er en antiderivert til $f(x)$.
Så alle kontinuerlige funksjoner har en antiderivert.)

Hvis $F(x)$ er en antiderivert til $f(x)$

$$I(x) = F(x) + C$$

$$I(a) = \int_a^a f(x)\,dx = 0$$

$$I(a) = 0 = F(a) + C$$

$$C = -F(a)$$

Så $$I(x) = F(x) - F(a)$$

$x=b$ gir

$$\boxed{I(b) = \int_a^b f(x)\,dx = F(b) - F(a)}$$

$$F(b) - F(a) = F(x)\Big|_a^b$$

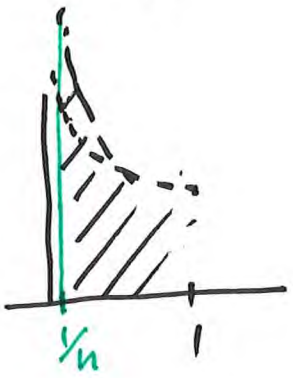
$$\int_0^1 \sqrt{x}\,dx =$$

$$\int_0^1 x^{1/2}\,dx = \frac{x^{3/2}}{3/2}\Big|_0^1$$

$$= \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}(1-0) = \frac{2}{3}$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\,\Big|_0^1$$

$$= 2(1-0) = \underline{2}$$



$$\lim_{n\to\infty} \int_{1/n}^1 \frac{1}{\sqrt{x}}\,dx$$

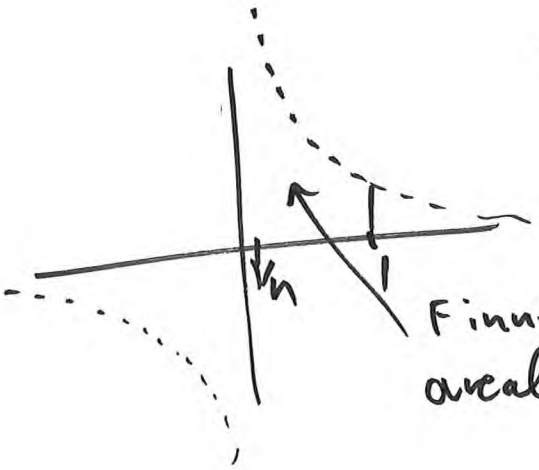
$$= \lim_{n\to\infty} \left(2 - 2\sqrt{\tfrac{1}{n}}\right) = \underline{\underline{2}}$$

$$\int_{-2}^1 \frac{1}{x}\,dx = \ln|x|\,\Big|_{-2}^1 \quad \text{Gir ikke mening.}$$

$$= \ln(1) - \ln(2)$$

$$= -\ln(2) \sim -0.69$$

Integralet finnes ikke!



$$\int_{1/n}^1 \frac{1}{x}\,dx$$

$$= \ln|x|\,\Big|_{1/n}^1 = \ln 1 - \ln\left|\tfrac{1}{n}\right|$$

$$= -\ln\left(\tfrac{1}{n}\right)$$

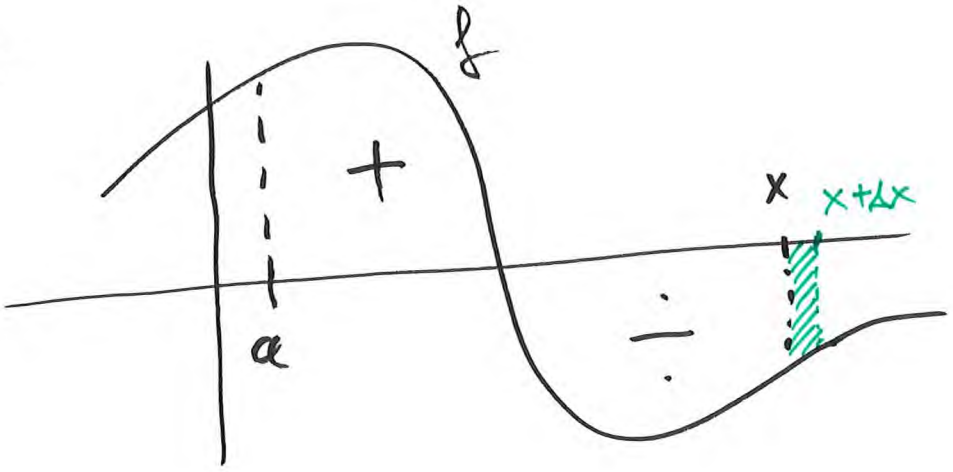
$$= -\ln(n^{-1})$$

$$= \underline{\ln(n)}$$

Så $\int_0^1 \frac{1}{x}\,dx = \lim_{n\to\infty} \int_{1/n}^1 \frac{1}{x}\,dx$

$$= \lim_{n\to\infty} \ln n = \infty$$

Bevisskisse



$$I(x) = \int_a^x f(t)\,dt$$

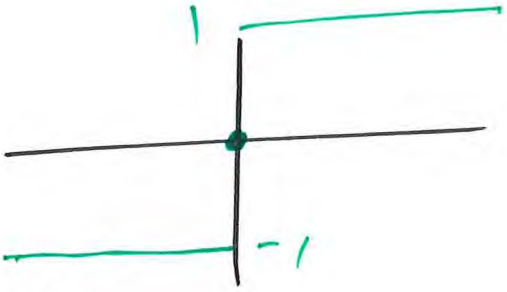
$$\frac{I(x+\Delta x) - I(x)}{\Delta x}$$

gjennomsnittlig verdi til $f(t)$ i $[x, x+\Delta x]$.

siden $f(x)$ er kontinuerlig

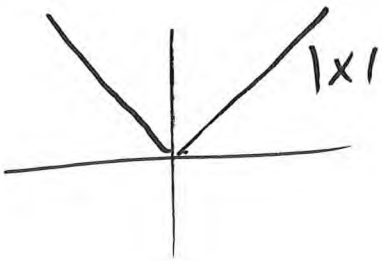
$$I'(x) = \lim_{\Delta x \to 0} \frac{I(x+\Delta x) - I(x)}{\Delta x} = f(x)$$

$$S(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



$S(x)$ har ikke en antiderivert.

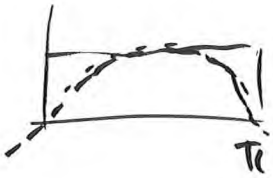
$$|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$



$$(|x|)' = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

ikke definert: $x = 0$.

$$\int_0^{\pi} \sin x \, dx$$

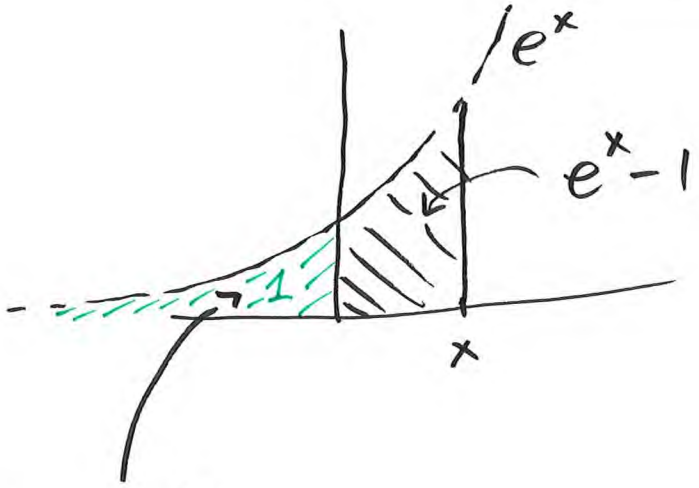


$$= -\cos x \Big|_0^{\pi}$$

$$= -\cos(\pi) - (-\cos(0))$$

$$= 1 + 1 = \underline{2}$$

$$\int_0^x e^t \, dt = e^t \Big|_0^x = \underline{e^x - 1}$$



$$\int_{-\infty}^{0} e^t \, dt$$

$$= \lim_{N \to \infty} \int_{-N}^{0} e^t \, dt = \lim_{N \to \infty} e^t \Big|_{-N}^{0}$$

$$= \lim_{N \to \infty} (1 - e^{-N}) = \underline{1}$$

$$\int_{-\infty}^{x} e^t \, dt = e^x$$

Linear substitusjo.

$$F'(x) = f(x)$$

$$\frac{d}{dx} F(ax+b) = F'(ax+b) \cdot \underbrace{(ax+b)'}_{a}$$

$$f(ax+b) \cdot a$$

$$\frac{d}{dx}\left(\frac{1}{a} F(ax+b)\right) = f(ax+b).$$

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

$$u = ax+b.$$

$$\int \sin(3x+1)\,dx \qquad u = 3x+1, \quad u' = 3$$

$$= \frac{1}{3}\int \sin(u)\,du$$

$$= \frac{1}{3}(-\cos(u)) + C$$

$$= \underline{\frac{-1}{3}\cos(3x+1) + C}$$

$$\int_0^1 (2x+3)^4\,dx \qquad u = 2x+3, \quad u' = 2$$

$$\int (2x+3)^4\,dx$$

$$= \frac{1}{2}\int u^4\,du = \frac{1}{2}\frac{u^5}{5} + C$$

$$= \underline{\frac{1}{10}(2x+3)^5 + C}$$

$$\int_0^1 (2x+3)^4 \, dx$$

$$= \frac{1}{10}(2x+3)^5 \Big|_0^1$$

$$= \frac{1}{10}(5^5 - 3^5) = \frac{2882}{10}$$

$$\int_a^b f(cx+d) \, dx$$

$$= \frac{1}{c} \int_{c\cdot a+d}^{cb+d} f(u) \, du$$

grensene endres til $u(a)$ og $u(b)$.

$\frac{1}{c}F(cx+d)$ er en antiderivert til $f(cx+d)$

Fundamentalteoremet:

$$\int_a^b f(cx+d) \, dx = \frac{1}{c} F(cx+d) \Big|_a^b$$

$$= \frac{1}{c} F(u) \Big|_{c\cdot a+d}^{c\cdot b+d}$$

$$= \frac{1}{c} \int_{c\cdot a+d}^{c\cdot b+d} f(u) \, du$$

oppg. $\int_{-1}^{2} (x+1)^3 \, dx$ substitusjon.

$u = x+1$

$u' = 1$

$u(-1) = 0$

$u(2) = 3$

$$\int_{u(-1)}^{u(2)} u^3 \, du$$

$$= \int_0^3 u^3 \, du$$

$$= \frac{u^4}{4} \Big|_0^3 = \frac{3^4 - 0}{4} = \frac{(3^2)^2}{4} = \frac{81}{4}$$

$$\int_0^{\pi/4} \cos(2x) \, dx$$

$u = 2x$

$u' = 2$

$du = 2dx$

$\frac{1}{2} du = dx$

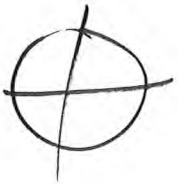
$$\int_{u(0)}^{u(\pi/4)} \cos(u) \, \frac{1}{2} du$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos u \, du$$

$$= \frac{1}{2} \sin(u) \Big|_0^{\pi/2}$$

$$= \frac{1}{2} \left(\underbrace{\sin \frac{\pi}{2}}_{1} - \underbrace{\sin(0)}_{0} \right)$$

$$= \frac{1}{2}$$



$$\int_1^2 \frac{1}{(3x-2)^2}\,dx$$

$u = 3x-2$

$u' = 3$

$du = 3dx$

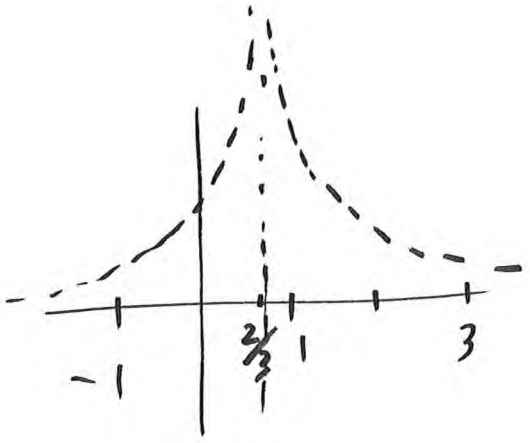
$\frac{1}{3}du = dx$

$$= \int_{u(1)}^{u(2)} \frac{1}{u^2}\cdot\frac{1}{3}\,du$$

$$= \frac{1}{3}\int_1^4 \frac{1}{u^2}\,du$$

$$= \frac{1}{3}\int_1^4 u^{-2}\,du = \frac{1}{3}\cdot\frac{u^{-1}}{-1}\Big|_1^4$$

$$= \frac{-1}{3}\left(\frac{1}{4}-\frac{1}{1}\right) = \frac{-1}{3}\left(\frac{1-4}{4}\right)$$

$$= \frac{1}{4}$$

oppg $\int_0^3 \frac{1}{(3x-2)^2}\,dx$ eksisterer ikke.