9 april
25

15E Integrasjonsmetoder

Substitusjon (variabelskifte)
Delvis integrasjon
Delbrøkoppspalting

(- kjerneregel
- produktregel)

$F'(x) = f(x)$ F antiderivert til f

Substitusjon

$$\frac{d}{dx} F(u(x)) = F'(u(x)) \cdot u'(x)$$

kjerneregel

$$\frac{d}{dx} F(u(x)) = f(u(x)) \cdot u'(x)$$

$$\boxed{\int f(u(x)) \cdot u' \, dx = F(u(x)) + C = \int f(u) \, du}$$

($u = x^2$ $u' = 2x$ så $x = \frac{1}{2}u'$)

$$\int x \sin(x^2) \, dx$$

$$= \int \frac{1}{2} u' \sin(u) \, dx = \frac{1}{2} \int \sin(u) \, du = \underline{-\frac{1}{2} \cos(x^2) + C}$$

$$\int \sin(x^2)\,dx = \int \frac{2x}{2x}\sin(x^2)\,dx$$

prøver med subst. $u=x^2$

$x=\sqrt{u}$ når $x \geq 0$

$$= \int \frac{1}{2x}\sin(\cancel{x}u)\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\sin(u)\,du$$

$$=$$

$*$ $$\int \frac{1}{x\ln|x|}\,dx = \int \underbrace{\left(\frac{1}{x}\right)}_{u'}\cdot \underbrace{\frac{1}{\ln|x|}}_{\frac{1}{u}}\,dx$$

benytter at $(\ln|x|)' = \frac{1}{x}$

$$\overset{\text{sub.}}{=} \int \frac{1}{u}\,du = \ln|u| + C = \underline{\ln|\ln|x|| + C}$$

$(\cos x)' = -\sin x$

$*$ $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$$= \int \underbrace{-(-\sin x)}_{u'}\cdot\underbrace{\frac{1}{\cos x}}_{\frac{1}{u}}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= \underline{-\ln|\cos x| + C}$$

$$u' = \frac{du}{dx}, \quad du = u'\,dx$$

$$\int f(u) \cdot u'\,dx = \int f(u) \frac{du}{dx}\,dx = \int f(u)\,du$$

$$u = 1+x^4$$
$$u' = 4x^3$$
$$x^3 = \frac{1}{4}u'$$
$$x^3\,dx = \frac{1}{4}u'\,dx = \frac{1}{4}du$$

$$\int x^3(1+x^4)^2\,dx = \int \frac{1}{4}u^2\,du = \frac{1}{4}\cdot\frac{u^3}{3} + C = \underline{\frac{1}{12}(1+x^4)^3 + C}$$

$$\int x^2(1+x^4)^2\,dx = \int x^2(1+2x^4+x^8)\,dx = \int x^2+2x^6+x^{10}\,dx = \underline{\frac{x^3}{3} + \frac{2x^7}{7} + \frac{x^{11}}{11} + C}$$

$$\int \sin^3 x \, dx \underset{\text{omskriving}}{=} \int \sin x \cdot \underbrace{\sin^2 x}_{1-\cos^2 x \text{ Pytagoras}} \, dx$$

$u = \cos x$

$u' = -\sin x$

$$= \int \underbrace{\sin x}_{-u'} \cdot \big(\underbrace{1 - \cos^2 x}_{1-u^2} \big) \, dx$$

$$\overset{\text{sub.}}{=} -\int 1 - u^2 \, du = \int u^2 - 1 \, du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C$$

Test: Deriverer $\left(\frac{1}{3} \cos^3 x - \cos x \right)'$

$$= \cos^2 x \cdot (\cos x)' - (\cos x)'$$

$$= \big(\underbrace{\cos^2 x - 1}_{-\sin^2 x \text{ pyt.}} \big) (- \sin x) = \underline{\underline{\sin^3 x}}$$

Delvis integrasjon

Produktregelen: $(u \cdot v)' = u' \cdot v + u \cdot v'$

$$u \cdot v + C = \int u' \cdot v + u \cdot v' \, dx$$

$$= \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

$$\boxed{\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' \, dx}$$

$$u = -\cos x$$

$$\int \underbrace{x}_{v} \cdot \underbrace{\sin x}_{u'} \, dx = x(-\cos x) - \int 1 \cdot (-\cos x) \, dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$\int x \sin x \, dx = \underline{-x\cos x + \sin x + C}$$

$$\int \underset{u'}{x} \cdot \underset{v}{\sin x}\, dx \qquad u = x^2/2$$

$$= \frac{x^2}{2} \sin x - \underbrace{\int \frac{x^2}{2} \cos x \, dx}_{\text{mer kompleks enn int. vi startet med.}}$$

$$u = x \qquad v' = \frac{1}{x}$$

$$\int \ln|x| \, dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln|x|}_{v} \, dx$$

$$\overset{\text{delvis}}{=} x \ln|x| - \int \underbrace{x \cdot \frac{1}{x}}_{1} \, dx$$

$$\int \ln|x| \, dx = \underline{x \ln|x| - x + C}$$

$$\int \underbrace{x^n}_{v} \underbrace{e^x}_{u'} \, dx = x^n e^x - \int n x^{n-1} e^x \, dx \qquad u = e^x, \quad v' = n x^{n-1}$$

Rekursiv formel

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx$$

$$= x^3 e^x - 3\left(x^2 e^x - 2\int x e^x \, dx\right)$$

$$= x^3 e^x - 3x^2 e^x + 6\left(x e^x - e^x\right) + C$$

$$= \left(x^3 - 3x^2\right) e^x + 6\left(x e^x - e^x\right) + C$$

$$= \underline{\left(x^3 - 3x^2 + 6x - 6\right) e^x + C}$$

$$u = \frac{-1}{2} e^{-2x}$$

$$\int \underbrace{e^{-2x}}_{u'} \underbrace{\sin(3x)}_{v} \, dx \overset{\text{delvis}}{=} \frac{-1}{2} e^{-2x} \sin(3x) - \int \underbrace{\frac{-1}{2} e^{-2x}}_{u'} \underbrace{(3\cos(3x))}_{v} \, dx$$

$$\overset{\text{delvis}}{=} \frac{-1}{2} e^{-2x} \sin(3x) + \frac{3}{2}\left[\frac{-1}{2} e^{-2x} \cdot \cos(3x) - \int \frac{-1}{2} e^{-2x} \cdot 3(-\sin(3x)) \, dx \right]$$

$$= \frac{-1}{2} e^{-2x} \sin(3x) - \frac{3}{4} e^{-2x} \cos(3x) - \left(\frac{3}{2}\right)^2 \int e^{-2x} \sin(3x) \, dx$$

"flytter over"

$$\underbrace{\left(1+\frac{9}{4}\right)}_{\frac{13}{4}} \int e^{-2x}\sin(3x)\,dx = \frac{-1}{2} e^{-2x}\sin(3x) - \frac{3}{4} e^{-2x}\cos(3x) + C$$

$$\int e^{-2x}\sin(3x)\,dx = \underline{\frac{4}{13} e^{-2x}\left(-\frac{\sin(3x)}{2} - \frac{3\cos(3x)}{4}\right) + C}$$

Delbrøksoppspalting

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$\frac{(x+1) - (x-1)}{2(x-1)(x+1)}$$

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right)dx$$

$$= \frac{1}{2}\left(\int \frac{1}{x+1}\,dx - \int \frac{1}{x-1}\,dx\right)$$

$$= \underline{\frac{1}{2}\left(\ln|x+1| - \ln|x-1|\right) + C}$$

$\int \frac{x}{x^2-1}\,dx$ her kan substitusjon benyttes

$u = x^2-1$

$u' = 2x$

$$\int \frac{x}{x^2-1}\,dx = \int \frac{1}{u}\,\frac{1}{2}\,du = \underline{\frac{1}{2}\ln|x^2-1| + C}$$

Alternativt: $\frac{x}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$.

$$I = \int \frac{1}{x^2-x-6}\,dx$$

Faktoriser x^2-x-6

$= (x-3)(x+2)$

$$\frac{1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$

$(A+B)x + 2A - 3B = 0 \cdot x + 1$

$A+B = 0$

$2A - 3B = 1$

$A = 1/5$

$B = -1/5$.

$$I = \int \frac{1}{5}\left(\frac{1}{x-3} - \frac{1}{x+2}\right)dx = \underline{\frac{1}{5}\left(\ln|x-3| - \ln|x+2|\right) + C}$$

$$\int \frac{1}{\sin x}\,dx = \int \frac{\sin x}{\sin^2 x}\,dx$$

$$= \int \underbrace{\sin x}_{-u'} \underbrace{\frac{1}{1-\cos^2 x}}_{\frac{1}{1-u^2}}\,dx$$

$u = \cos x$

$u' = -\sin x$

$$\overset{\text{sub.}}{=} \int -1 \cdot \frac{1}{1-u^2}\,du = \int \frac{1}{u^2-1}\,du$$

$$= \frac{1}{2}\left(\ln|u+1| - \ln|u-1|\right) + C$$

$$\int \frac{1}{\sin x}\,dx = \underline{\frac{1}{2}\left(\ln|\cos x+1| - \ln|\cos x-1|\right) + C}$$