18 mars 2015

substitusjon (variabelbytte)

(1)

$$\int f(u(x)) \cdot u'(x)\, dx = \int f(u)\, du$$

eksempel

$$\int e^{x^2} \cdot 2x\, dx =$$

$$\int e^{u(x)} \cdot u'(x)\, dx = \int e^u\, du =$$

$u = x^2 \quad u' = 2x \quad \Big| \quad e^u + c = \underline{e^{x^2} + c}$

bevis:

La $F(x)$ være en antiderivert til $f(x)$

$$\frac{d}{dx} F(u(x)) = \frac{dF}{du} \cdot \frac{du}{dx} = f(u(x)) \cdot u'(x)$$

kjerneregel

så $F(u(x))$ er en antiderivert til $f(u(x)) \cdot u'(x)$

$$\int f(u(x))\, u'(x)\, dx = F(u(x)) + c$$

$$= \int f(u)\, du \quad \left(\begin{array}{l}\text{hvor } u \\ \text{erstattes} \\ \text{av } u(x)\end{array}\right)$$

$\left(u'(x)dx = \frac{du}{dx} dx = du\right)$

eksempler: $I = \int \overbrace{\sin(x)}^{-u'} \cos(\overbrace{\cos(x)}^{u})\, dx$

$u(x) = \cos x \quad u'(x) = -\sin x$ så $-u'(x) = \sin(x)$

$$I = \int -u'(x) \cos(u(x))\, dx = -\int \cos(u)\, du$$

$$= -\sin(u) + c = \underline{-\sin(\cos(x)) + c}$$

② e^{x^2} har en antiderivert (kont. funksjon ...) men den er <u>ikkje</u> en elementær funksjon (bygd opp av de "vanlige" funksjonene)

$$\int \underbrace{2\sin x}_{u} \underbrace{\cos x}_{u'}\, dx = \int 2u\, du$$
$$= u^2 + C$$
$$= \sin^2(x) + C_1$$

$$\int \underbrace{2\sin x}_{-v'} \underbrace{\cos x}_{v}\, dx = \int -2 \cdot v\, dv$$
$$= -v^2 + C$$
$$= -\cos^2 x + C_2.$$
$$= 1 - \cos^2 x + (C_2 - 1)$$
$$= \sin^2(x) + (C_2 - 1)$$

Linear substitusjon $\quad u = ax + b$

$u' = a \quad (\neq 0)$

$$\int f(ax+b)\, dx = \frac{1}{a} F(ax+b) + C$$

hvor $F(x)$ er en antiderivert til $f(x)$

(3) $\int x\,\underbrace{(x-1)^3(x+1)^3}\,dx$

$$\int \underbrace{x}_{\frac{1}{2}u'}\,(\underbrace{x^2-1}_{u})^3\,dx$$

$$= \int \tfrac{1}{2}u'\,u^3\,dx = \tfrac{1}{2}\int u^3\,du$$

$$= \tfrac{1}{2}\,\frac{u^4}{4} + C = \underline{\tfrac{1}{8}(x^2-1)^4 + C}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$x = \tan y = \frac{\sin(y)}{\cos(y)} \quad ,(y = \arctan x)$$

$$\frac{dx}{dy} = 1+\tan^2 y \quad \text{(sjekk dette!)}$$

$$= 1+x^2.$$

$$\frac{dy}{dx}\cdot\frac{dx}{dy} = \frac{d\,y(x(y))}{dy} = 1$$

$$\frac{d\arctan x}{dx} = \frac{dy}{dx} = \frac{1}{dx/dy} = \underline{\frac{1}{1+x^2}}.$$

$$\int \frac{1}{(2x+1)^2+1}\,dx = \tfrac{1}{2}\arctan(2x+1) + C$$

$$= \int \frac{1}{4x^2+4x+2}\,dx \quad \text{(her har vi ganget ut } (2x+1)^2\text{)}$$

(4) $$\int \frac{13}{4x^2+4x+10}\,dx = \int \frac{13}{(2x+1)^2+9}\,dx$$

$u = 2x+1 \qquad u' = 2$

$$\int \frac{13}{u^2+9} \cdot \frac{1}{2}\,du$$

$$\int \frac{13}{9\left(\frac{u^2}{3^2}+1\right)} \frac{1}{2}\,du$$

prøver med $\quad v = \frac{u}{3} \qquad v' = \frac{dv}{du} = \frac{1}{3}$

$3\,dv = du$

$$\int \frac{13}{9(v^2+1)} \cdot \frac{1}{2} \cdot 3\,dv = \frac{13 \cdot 3}{2 \cdot 9} \int \frac{1}{v^2+1}\,dv$$

$$= \frac{13}{6}\arctan(v) + c = \frac{13}{6}\arctan\left(\frac{u}{3}\right) + c$$

$$= \frac{13}{6}\arctan\left(\frac{2x+1}{3}\right) + c$$

$$\int \frac{1}{4x^2+4x+1}\,dx = \int \frac{1}{(2x+1)^2}\,dx$$

$u = 2x+1 \qquad \frac{du}{dx} = 2 \qquad \frac{1}{2}du = dx$

$$\int \frac{1}{u^2}\frac{1}{2}\,du = \frac{1}{2}\int u^{-2}\,du = \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + c$$

$$= \frac{-1}{2u} + c = \frac{-1}{2(2x+1)} + c$$

(5) $\displaystyle\int \frac{3}{4x^2+4x}\,dx = \frac{3}{4}\int \frac{1}{x^2+x}\,dx$

$x^2+x = x(x+1)$

$$\frac{1}{x(x+1)} = \frac{A}{x+1} + \frac{B}{x}$$

$$\frac{1}{x(x+1)} = \frac{x\cdot A + B(x+1)}{x(x+1)}$$

så $0\cdot x + 1 = x(A+B) + B$

så $B = 1$, $A+B=0$, $A=-B=-1$.

$$\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x} - \frac{1}{x+1}\,dx$$

$$= \ln|x| - \ln|x+1| + c$$

integralet ~~vårt~~ er lik: $\underline{\frac{3}{4}\left(\ln\left|\frac{x}{x+1}\right|\right) + c}$

Vi kan finne en antiderivert som er en elementær funksjon til alle rasjonale.
se 6.4 i boka.

$$\int \frac{2}{x^2-1}\,dx = \int \frac{2}{(x-1)(x+1)}\,dx$$

Delbrøksoppspalting: $\displaystyle\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$

$$\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \ln|x-1| - \ln|x+1| + c$$

$$= \underline{\ln\left|\frac{x-1}{x+1}\right| + c}$$

(6) $\int \frac{1}{(x^2+1)(2x-1)}\,dx$

$$\frac{1}{(x^2+1)(2x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{2x-1}$$

Finner felles nevner $1 = (Ax+B)(2x-1) + C(x^2+1)$

$$x^2(2A+C) + x(2B-A) + (-B+C) = 1$$

$C - B = 1$

$A = 2B$, $B = \frac{1}{2}A$

$C = -2A$

$$\overset{C}{(-2A)} - (\tfrac{1}{2}A) = 1$$

$$-\frac{5}{2}A = 1$$

så $A = \frac{-2}{5}$, $B = \frac{1}{2}A = \frac{-1}{5}$ $C = -2A = \frac{4}{5}$.

int. er lik $\frac{1}{5}\int \frac{-2x-1}{x^2+1} + \frac{4}{2x-1}\,dx$

$$= \frac{1}{5}\left[-\int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx + 4\int \frac{1}{2x-1}\,dx\right]$$

$u = x^2+1$

$u' = 2x$

linear substitusjon

$$= \frac{1}{5}\left[-\ln(x^2+1) - \arctan(x) + 2\ln|2x-1|\right] + c$$

polynom divisjon

$$\int \frac{x^2+3}{x-1}\,dx = \int x+1 + \frac{4}{x-1}\,dx$$

$$= \frac{x^2}{2} + x + 4\ln|x-1| + c$$

⑦ $\int \frac{1}{(1+x^2)^2} dx = \int \frac{(1+x^2)-x^2}{(1+x^2)^2} dx$

$= \int \frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2} dx$

$= \arctan x - \int \underbrace{x}_{v} \cdot \underbrace{\frac{x}{(1+x^2)^2}}_{u'} dx$

(delvis integrasjon)

$u = \frac{-1}{2}\left(\frac{1}{1+x^2}\right)$

$v' = 1$

$= \arctan x - \left[\frac{(-1/2)\cdot x}{1+x^2} - \int \underset{v'}{1} \cdot \underset{u}{\frac{-1}{2} \frac{dx}{1+x^2}} \right]$

$= \arctan x + \frac{x}{2(1+x^2)} - \frac{1}{2} \underbrace{\int \frac{1}{1+x^2} dx}_{\arctan x}$

$= \underline{\frac{1}{2} \arctan x + \frac{x}{2(1+x^2)} + c}$

$u = \frac{1}{a} e^{ax}$

$v' = b\cos(bx)$

$\int \overbrace{e^{ax}}^{u'} \overbrace{\sin(bx)}^{v} dx$

delvis integrasjon en gang til.

$= \frac{1}{a} e^{ax} \sin(bx) - \int \underbrace{\frac{1}{a} e^{ax}}_{u'} b \underbrace{\cos(bx)}_{w} dx$

$= \frac{1}{a} e^{ax} \sin(bx) - \left[\frac{b}{a} \underset{u}{\left(\frac{1}{a} e^{ax}\right)} \underset{v}{\cos(bx)} - \int \underset{u}{\frac{b}{a^2} e^{ax}} \underset{v'}{(-b\sin bx)} dx \right]$

$= \frac{e^{ax}\sin(bx)}{a} - \frac{b e^{ax}\cos(bx)}{a^2} - \frac{b^2}{a^2} \underbrace{\int e^{ax} \sin(bx) dx}_{\text{samme integral vi starta med.}}$!

Flytter integralen over til venstre side

$\left(1+\frac{b^2}{a^2}\right) \int e^{ax} \sin(bx) dx = \frac{a e^{ax} \sin(bx) - b e^{ax} \cos(bx)}{a^2}$

deler med $\frac{a^2+b^2}{a^2}$

$\int e^{ax} \sin(bx) dx = \frac{(a\sin(bx) - b\cos(bx)) e^{ax}}{a^2+b^2}$.