Lineære transformasjoner

$$\vec{x} \qquad\qquad\qquad \vec{y}$$

$$\mathbb{R}^n \xrightarrow{\text{transformasjon}} \mathbb{R}^m$$

$$\underbrace{\vec{y}}_{\in \mathbb{R}^m} = \underbrace{M}_{\text{Standardmatrisen til transformasjonen } \in \mathbb{R}^{m\times n}} \overbrace{\vec{x}}^{\in \mathbb{R}^n}$$

Lineære transformasjoner tilfredsstiller:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

2. $T(k\vec{u}) = kT(\vec{u})$

3. $T(\vec{v}) = M\vec{v}$

Enhetsvektorer

$$M = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

f.eks: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Eksempel:

Regn ut standardmatrisen M når

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}3\\1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\\2\end{bmatrix}.$$

Løsning:

$$3\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix}$$

$$\frac{1}{5}\left(3\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}\right) = \frac{1}{5}\left(\begin{bmatrix}3\\6\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} = \vec{e}_2$$

Kolonne 2 i M:

$$T(\vec{e}_2)$$

$$= T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\underset{\text{"}k\text{"}}{\frac{1}{5}}\left(3\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}\right)\right)$$

$T(k\vec{u}) = kT(\vec{u})$

$$= \frac{1}{5}T\left(3\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}\right) = \frac{1}{5}\left(T\left(3\begin{bmatrix}1\\2\end{bmatrix}\right) - T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)\right)$$

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

$$= \frac{1}{5}\left(3T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)\right)$$

$$= \frac{1}{5}\left(3\cdot\begin{bmatrix}3\\1\\0\end{bmatrix} - \begin{bmatrix}0\\2\\2\end{bmatrix}\right) = \frac{1}{5}\underline{\begin{bmatrix}9\\1\\-2\end{bmatrix}}$$

Kolonne 1 i M:

$$\begin{bmatrix}1\\2\end{bmatrix} - 2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1-6\\2-2\end{bmatrix} = -5\begin{bmatrix}1\\0\end{bmatrix}$$

$$-\frac{1}{5}\left(\begin{bmatrix}1\\2\end{bmatrix} - 2\begin{bmatrix}3\\1\end{bmatrix}\right) = \quad \cdots \quad = \begin{bmatrix}1\\0\end{bmatrix} = \vec{e}_1$$

$$T(\vec{e}_1) = T\left(-\frac{1}{5}\left(\begin{bmatrix}1\\2\end{bmatrix} - 2\begin{bmatrix}3\\1\end{bmatrix}\right)\right)$$

$$= -\frac{1}{5} T\left(\begin{bmatrix}1\\2\end{bmatrix} - 2\begin{bmatrix}3\\1\end{bmatrix}\right)$$

$$= -\frac{1}{5}\left(T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - T\left(2\begin{bmatrix}3\\1\end{bmatrix}\right)\right)$$

$$= -\frac{1}{5}\left(T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - 2T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)\right)$$

$$= -\frac{1}{5}\left(\begin{bmatrix}3\\1\\0\end{bmatrix} - 2\begin{bmatrix}0\\2\\2\end{bmatrix}\right)$$

$$= -\frac{1}{5}\left(\begin{bmatrix}3-0\\1-4\\0-4\end{bmatrix}\right) = -\frac{1}{5}\begin{bmatrix}3\\-3\\-4\end{bmatrix}$$

Vi får:

$$M = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5}\begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix} & \frac{1}{5}\begin{bmatrix} 9 \\ 1 \\ -2 \end{bmatrix} \end{bmatrix}$$

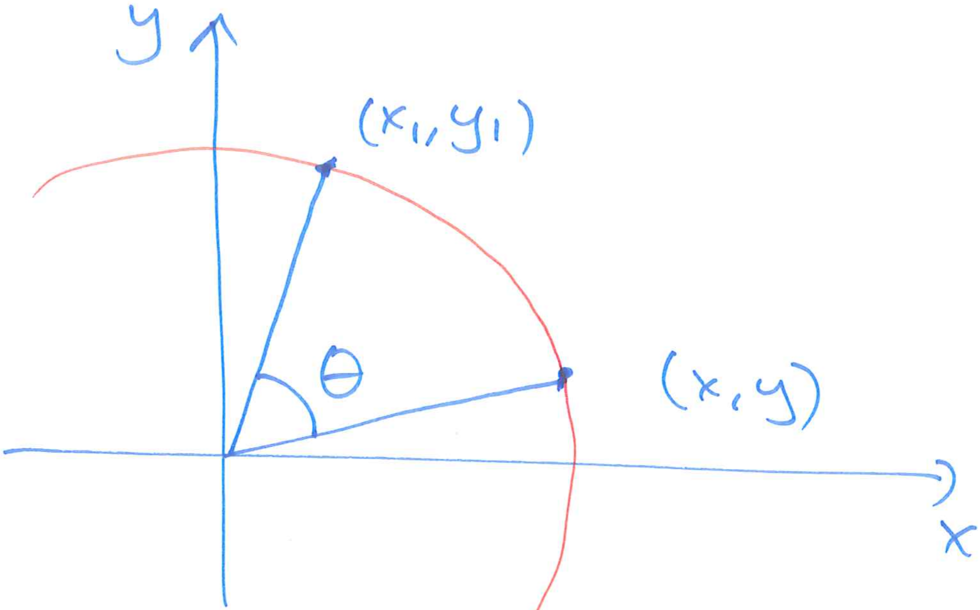
$$= \frac{1}{5}\begin{bmatrix} -3 & 9 \\ 3 & 1 \\ 4 & -2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \underline{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}.$$ Tester:

$$M\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 9 \\ 3 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3\cdot 1 + 9\cdot 2 \\ 3\cdot 1 + 1\cdot 2 \\ 4\cdot 1 - 2\cdot 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 15 \\ 5 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}$$ Stemmer!

Rotasjonsmatriser



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Rotasjonsmatrisen}} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$T(\vec{x}) = M\vec{x}$$

For eksempel: $\theta = \frac{\pi}{2}$

$$\Rightarrow M = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Sammensatte transformasjoner

To transformasjoner S, T.

$$S(T(\vec{u})) = (S \circ T)(\vec{u})$$

T har standardmatrise M

S ———"——— P

$$T\vec{u} = M\vec{u}, \quad S\vec{v} = P\vec{v}$$

$$\begin{aligned}(S \circ T)(\vec{u}) &= S(T(\vec{u})) \\ &= S(M\vec{u}) \\ &= \underbrace{P \cdot M}\vec{u}\end{aligned}$$

Standardmatrisen til $S \circ T$.

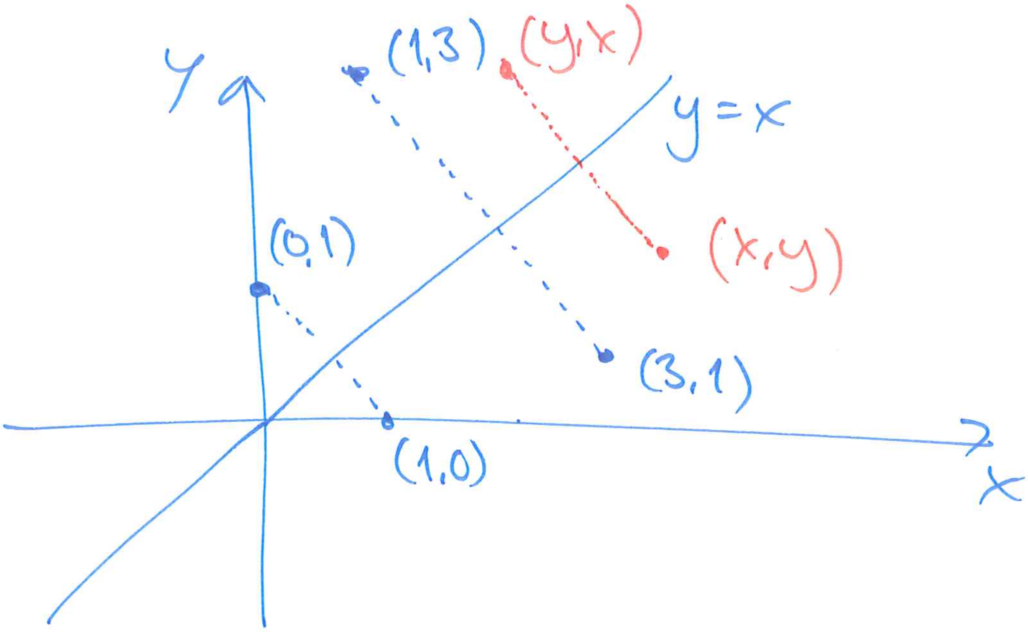
Generelt: $S \circ T \neq T \circ S$ fordi

generelt: $PM \neq MP$.

Eksamensoppgave: Oppg. 3 des 2015

En lineær transformasjon $T:\mathbb{R}^2 \to \mathbb{R}^2$ er gitt ved speiling om $y=x$.

Bestem standardmatrisen.



T tar vektoren $\begin{bmatrix}x\\y\end{bmatrix}$ og bytter koordinater til $\begin{bmatrix}y\\x\end{bmatrix}$. Da blir standardmatrisen:

$$M\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}y\\x\end{bmatrix}$$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cancel{ax}+by\\cx+\cancel{dy}\end{bmatrix}$$

$$\Rightarrow \underline{M=\begin{bmatrix}0&1\\1&0\end{bmatrix}}$$

Kontinuitet og deriverbarhet

f er kontinuerlig i $x = a$ dersom

$$\lim_{x \to a} f(x) = f(a).$$

(spesielt må $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$).

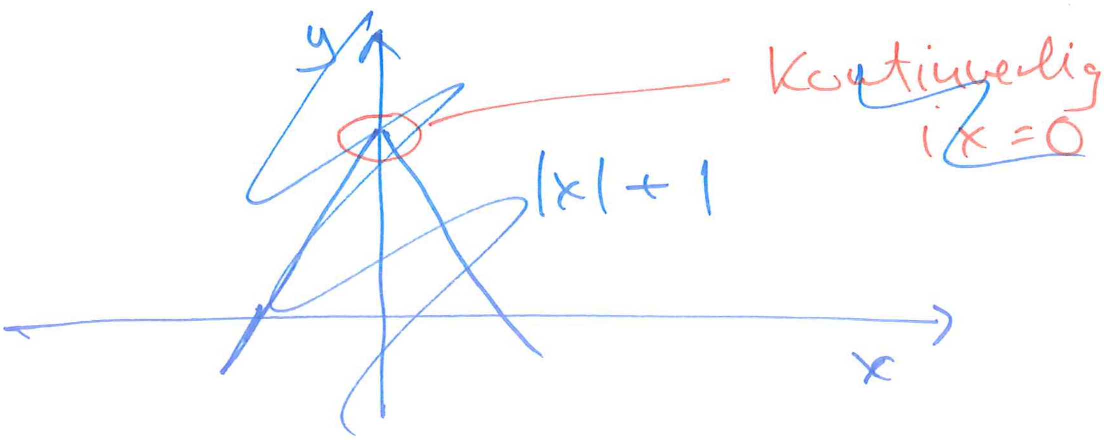
f er deriverbar i $x = a$ dersom

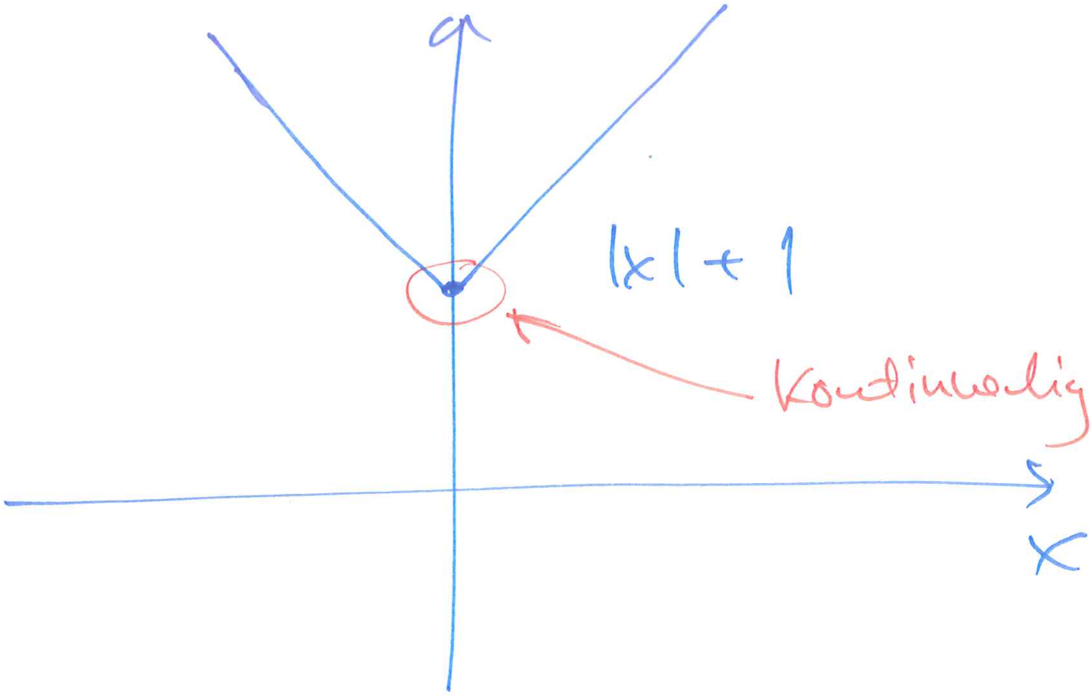
$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} \text{ eksisterer!}$$

Eksempel:

$$f(x) = |x| + 1 = \begin{cases} -x+1, & x < 0 \\ x+1, & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$





<u>Ikke deriverbar</u> i $x=0$ fordi:

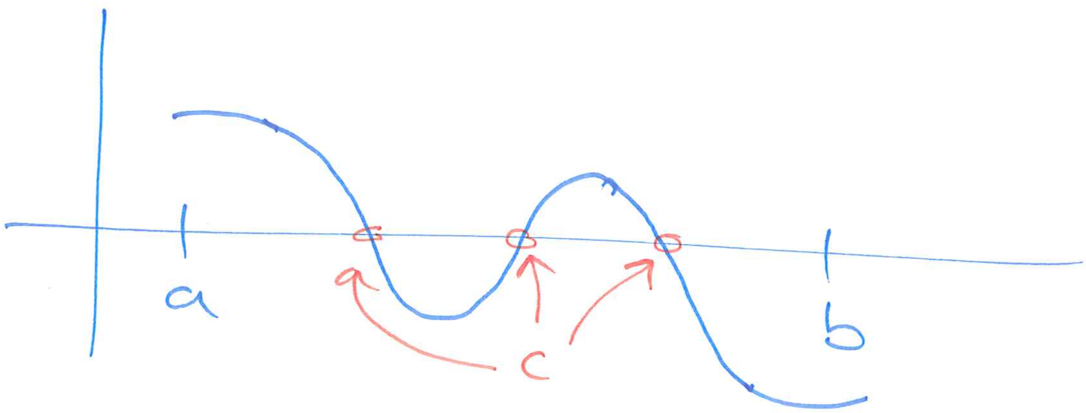
$$\lim_{\Delta x \to 0^+} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \frac{\Delta x + \cancel{1} - \cancel{1}}{\Delta x} = 1$$

$$\neq$$

$$\lim_{\Delta x \to 0^-} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \frac{-\Delta x + \cancel{1} - \cancel{1}}{\Delta x} = -1$$

Skjæringssetningen

f er kontinuerlig på $[a,b]$. Hvis $f(a)$ og $f(b)$ har ulikt fortegn, så finnes $c \in \langle a,b \rangle$ slik at $f(c) = 0$.



Eksempel:

$f(x) = xe^{x} - 2$.

Vis at f har nullpunkt på $\langle 0,2 \rangle$.

Hvordan vet vi at det bare er ett nullpunkt?

Løsning:

f er kontinuerlig på $[0,2]$.

$f(0) = 0 \cdot e^0 - 2 = -2 < 0$

$f(2) = 2e^2 - 2 = 2(e^2 - 1) > 0$

Da finnes $c \in \langle 0,2 \rangle$ slik at $f(c) = 0$.

$$f'(x) = (xe^x - 2)' = x \cdot e^x + 1 \cdot e^x$$
$$= e^x(x+1)$$

Dvs $f'(x) > 0$ på $[0,2]$

f er altså strengt voksende på $[0,2]$.

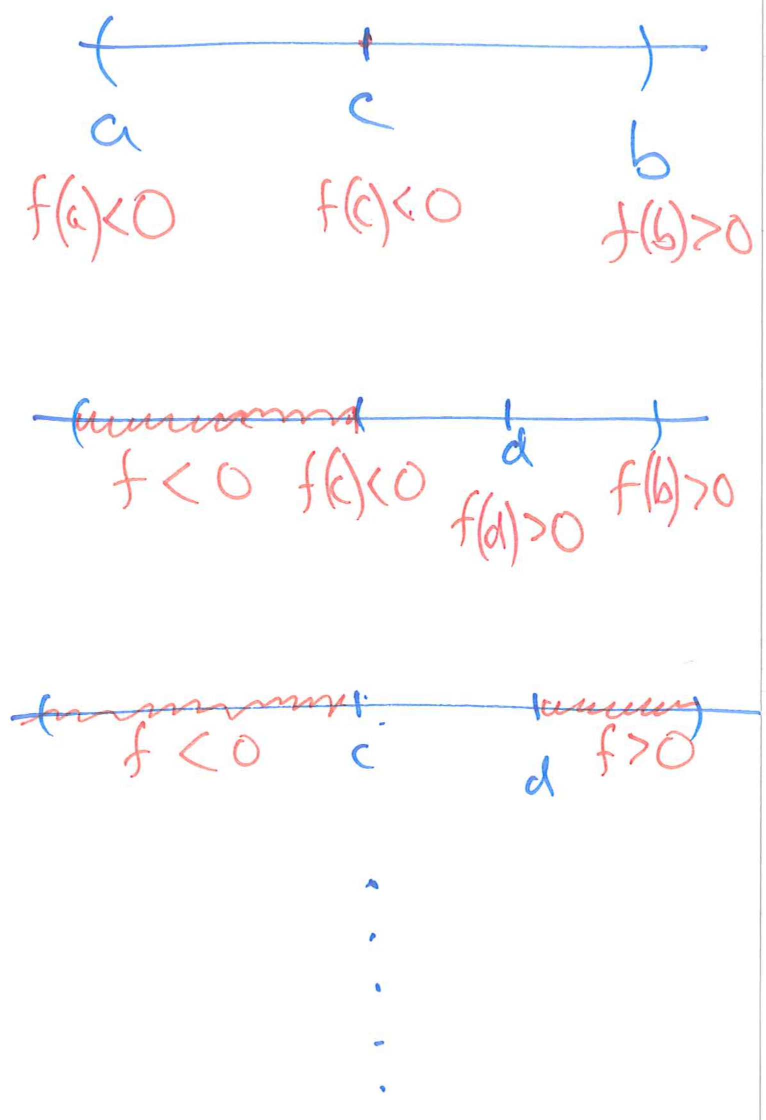
Kun ett nullpunkt på $\langle 0,2 \rangle$.

$$xe^x - 2 = 0$$

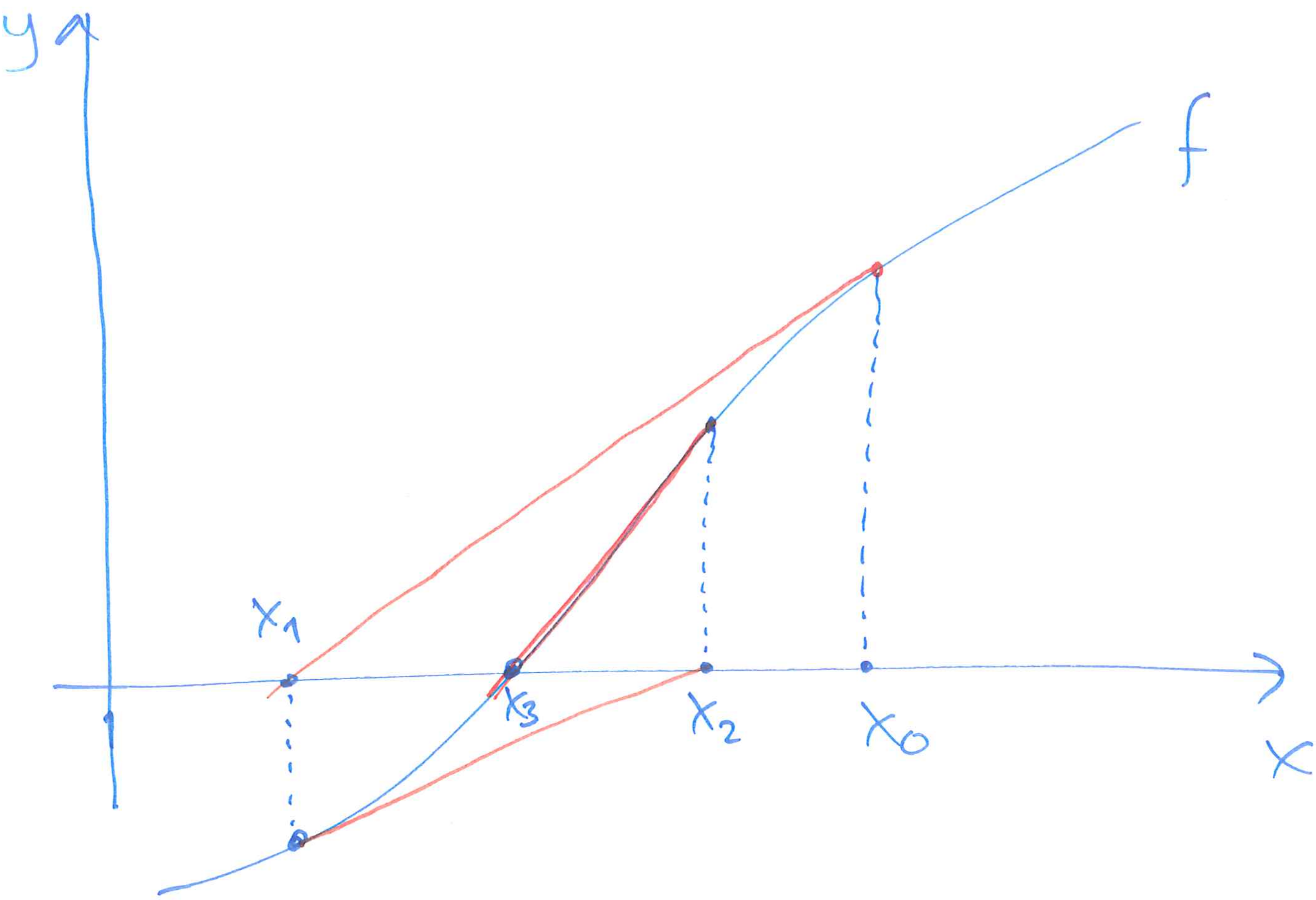
Kan ikke løses analytisk!

Halveringsmetoden

Starter med $[a,b]$ hvor vi vet det finnes et nullpunkt (fordi $f(a)$ og $f(b)$ har ulikt fortegn).



Newtons metode



Kan vises at

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Newtons metode

Eksempel:

$$f(x) = xe^x - 2, \quad x_0 = 2$$

$$f'(x) = e^x(x+1)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{2e^2 - 2}{e^2(2+1)} \approx \underline{1{,}4236}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1{,}0350$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0{,}8750$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx \underline{0{,}8530}$$

$$\approx$$

"Virkelig" verdi $\approx \underline{0{,}8526}$

Grenseverdier

Når f er definert for $x=a$ og f er kontinuelig i $x=a$ så er $\lim\limits_{x\to a} f(x) = f(a)$.

Ellers, f. eks. når vi får "$\frac{0}{0}$", "$\infty-\infty$", "$\frac{\infty}{\infty}$" eller "$0\cdot\infty$"-uttrykk må vi gjøre noe annet.

Eksempler:

1. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-x\right) = \lim\limits_{x\to\infty}\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x}$

$$= \lim_{x\to\infty}\frac{\cancel{x^2}+x-\cancel{x^2}}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}$$

$$\overset{\text{l'Hopital}}{=} \lim_{x\to\infty}\frac{1}{\frac{2x+1}{2\sqrt{x^2+x}}+1} = \frac{1}{1+1} = \underline{\underline{\frac{1}{2}}}$$

2. $$\lim_{x\to 0} \frac{\cos^2 x - 1}{\sin x} = \text{"}\frac{0}{0}\text{"}$$

$$\overset{\text{l'Hopital}}{=} \lim_{x\to 0} \frac{2\cancel{\cos x}\cdot(-\sin x)}{\cancel{\cos x}}$$

$$= -2 \lim_{x\to 0} (\sin x) = \underline{\underline{0}}$$

Alternativt:

$$\boxed{\cos^2 x + \sin^2 x = 1}$$

$$\Leftarrow \quad \cos^2 x - 1 = -\sin^2 x$$

$$\frac{\cos^2 x - 1}{\sin x} = -\frac{\sin^2 x}{\sin x} = \underline{-\sin x}$$